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THE ASTROPHYSICAL JOURNAL

AN INTERNATIONAL REVIEW OF SPECTROSCOPY AND ASTRONOMICAL PHYSICS

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VOLUME LX OCTOBER 1924 NUMBER 3

THE EMISSION SPECTRUM OF WATER-VAPOR

By WILLIAM W. WATSON

ABSTRACT

The origin of the ultra-violet band spectrum of water-vapor.—New experiments suggest that the carrier of these bands is the *hydroxyl ion* rather than the water molecule. Furthermore, the application of the quantum theory of band spectra to the fine structure of the λ 2811 and λ 3064 bands gives information favoring *OH* as the carrier.

Wave-lengths and series relations of lines of the λ 2811 band.—This band has been photographed with high dispersion and wave-lengths accurate to 0.01 *A* have been determined for all its lines. Every line has been assigned to a *parabolic series.* These series, together with those for the λ 3064 band as determined by Heurlinger, have been analyzed on the basis of the quantum theory of band spectra.

The moment of inertia of the carrier.—The two bands give the same final moment of inertia of 1.63×10^{-40} gr·cm², but different initial moments of inertia. Also the value of the final moment of inertia varies with m, the rotation quantum, while the initial moment of inertia is a constant, independent of the rotational velocity. The relation $R(m)-Q(m)=Q(m-1)-P(m)$, which should hold true for every value of m, is found to be valid only for that m which forms the turning-point of the R branch.

New band at λ 3021.—Fifteen new lines extending to the violet from the head of the λ 3064 band have been measured. These lines form a single parabolic series. Their carrier is unknown.

It is well known that with sources such as Geissler tubes containing water-vapor, moist hydrogen or oxygen, etc., sparks under water, and the oxy-hydrogen flame a beautiful, well-developed spectrum of the band type is obtained. This spectrum extends throughout the entire ultra-violet region, and has in the past been assigned to the water molecule as its carrier. Some new experiments presented in this paper, however, lead to the conclusion that these bands are due rather to the hydroxyl ion. Several hundred

lines have been accurately measured, and it is shown that the application of the quantum theory of band spectra to the fine structure of the bands yields results supporting this view.

The first comprehensive measurements of this spectrum were made by Liveing and Dewar,[1] with the oxy-hydrogen flame as the source. They found bands extending from λ 2268 in the ultra-violet to λ 4100 on the red side, with the main bands starting at λ 3064 and λ 2811. The λ 3064 band has been measured with high dispersion by Meyerheim[2] and by Grebe and Holtz.[3] In addition to the sources mentioned, these bands are found in the flames of hydrogen compounds, and in fact they appear whenever there is water-vapor in any source.

The author, after considerable experimental work, found the following method to yield the best spectra. A continuous flow of water-vapor was passed through a discharge tube of heavy pyrex glass which had a 5 mm bore and large cylindrical aluminum electrodes. One end was sealed with a quartz window and the other communicated with a tube filled with distilled water. No stopcocks were used, thereby eliminating the *CO* bands which usually put in an appearance if any stopcock grease is present. It was found necessary to flow the gas to do away with the continuous spectrum of hydrogen, which, when present in long exposures, obliterated the weaker lines. A strong uncondensed discharge was employed to excite this source. With this method excellent photographs of the weaker λ 2811 band have been obtained in the second order of a 21-foot concave grating in six to eight hours.

Grebe and Holtz[4] found that the presence of both oxygen and hydrogen was necessary for the production of this spectrum. When hydrogen was introduced into an oxygen tube, the intensity of the bands was increased tremendously, their faint appearance before the intermixture with hydrogen being due no doubt to the fact that the electrodes gave off a little hydrogen. If either the oxygen or the hydrogen were removed from the mixture, there was a marked reduction in the intensity of the bands.

[1] *Philosophical Transactions*, **179**, 27, 1888.

[2] *Zeitschrift für Wiss. Phot.*, **2**, 131, 1904.

[3] *Annalen der Physik*, **39**, 1243, 1912.

[4] *Loc. cit.*

Fowler[1] has shown that at least 150 solar lines in the region of the λ 3064 band are in close agreement with wave-lengths in Grebe and Holtz's measurements of this band. These Fraunhofer lines originate in the sun's atmosphere and have no connection with the so-called "rain bands," which are undoubtedly due to water-vapor in the earth's atmosphere.

Further evidence of a qualitative nature as to the origin of the bands was sought before the examination of their fine structure was begun. Water-vapor was flowed through a large-bore tube at a pressure of 2 mm (estimated) and excited by a weak electrodeless discharge. Even though the voltage was reduced to the point where the bands were obtained only after a 30-minute exposure with a quartz spectrograph, *the series lines* α, β, *and* γ *of hydrogen were still present*, thus indicating that hydrogen atoms were being ejected from the H_2O molecules. It should be mentioned that when benzene vapor is excited in this manner, one easily obtains the characteristic bands due to benzene in the region from λ 2700 to λ 3200, first described by Stewart and Marsh[2] in 1923. There is no indication that the relatively complicated benzene molecule is being "broken up." We thus see that even with a very small amount of energy, the H_2O molecules tend to break up at least into H and OH.

Next a very powerful disruptive discharge was sent through the heavy pyrex tube containing water-vapor at 20 mm pressure. The energy was so great that even this tube became overheated in a few seconds. In this case the characteristic oxygen spectrum made its appearance with such intensity that 15 seconds sufficed for the exposure time with the quartz spectrograph. The atomic hydrogen spectrum was of course very intense, and the ultra-violet band spectrum in question was of about the same intensity and character as with the uncondensed discharge through flowing water-vapor. But the disruptive discharge had evidently broken up a considerable percentage of the H_2O molecules into O and H.

The assumption that the carrier of these bands is the OH ion is then a plausible one in the light of the foregoing experiments. We are, therefore, interested in applying the quantum theory of band spectra due to dipoles to the parabolic series in these bands.

[1] *Proceedings of the Royal Society*, A, **94**, 472, 1918. [2] *Nature*, **111**, 115, 1923.

A number of investigators have worked on the series relationships between the lines of the λ 3064 band, chief among which are Meyerheim,[1] Fortrat,[2] Deslandres and D'Azambuja,[3] and Heurlinger.[4] None of these had our present quantum theory of the production of band spectra to apply to their results. Heurlinger,

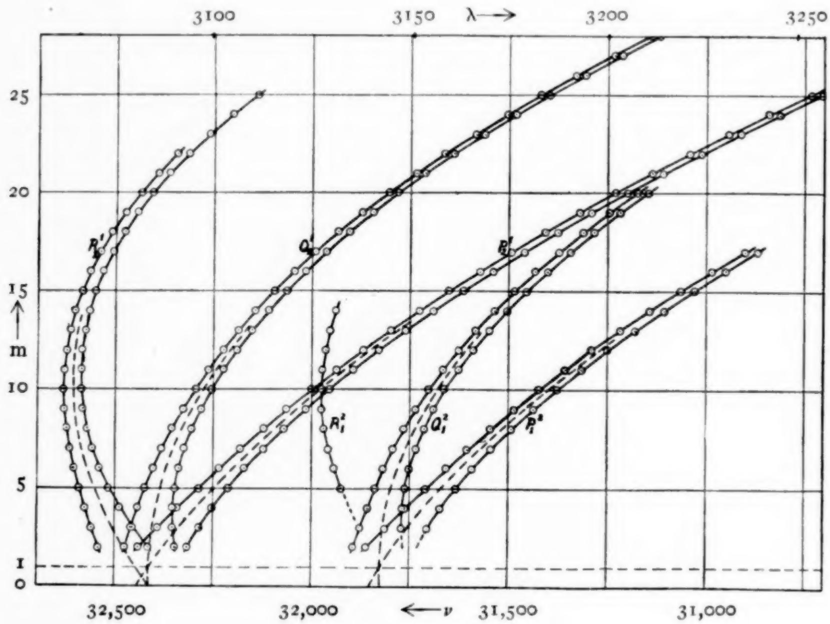


FIG. 1.—λ 3064 band

however, succeeded in assigning all but 10 of the 270 lines of Grebe and Holtz's measurements of the λ 3064 band into six parabolic series, P_1^p, Q_1^p, R_1^p, P_2^p, Q_2^p, R_2^p, where p has the values 1 or 2. A number of empirical relationships between the corresponding lines of the various series are set forth in his work. Figure 1 is a plot of Heurlinger's series for this band, as given on page xxx of his dissertation.

It is seen at a glance that the branches P_1, Q_1, R_1, or P_2, Q_2, R_2—and consequently the mean values indicated by the dotted lines—

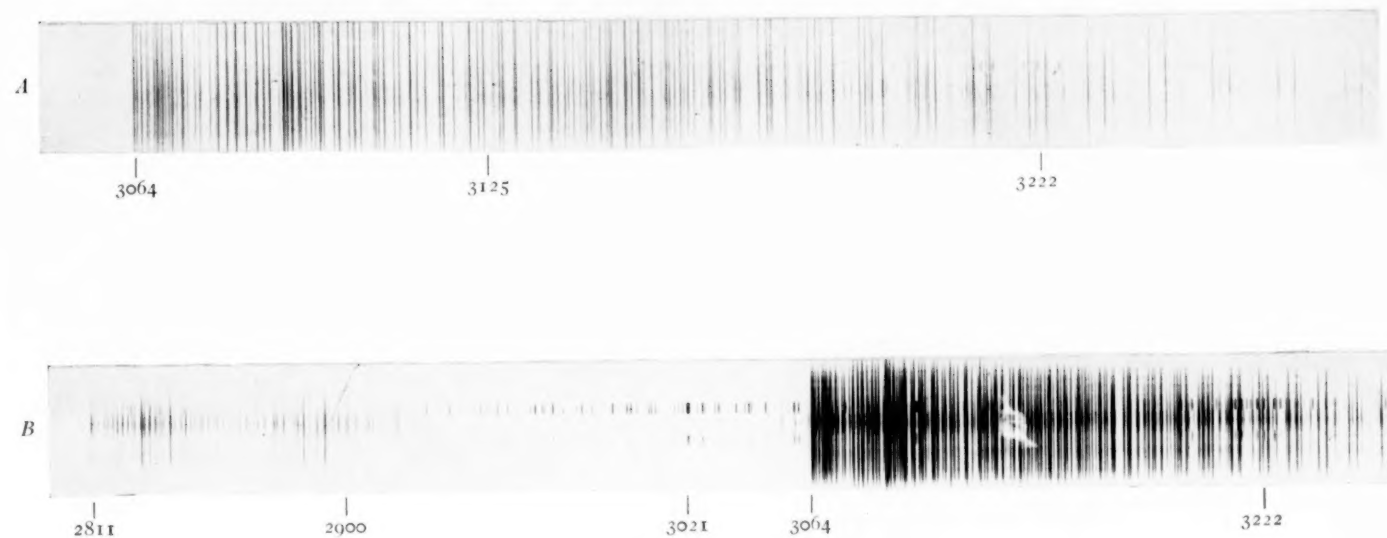
[1] *Loc. cit.*

[2] *Annales de Physique*, **3**, 282, 1915.

[3] *Comptes Rendus*, **157**, 814, 1913.

[4] Lund, Dissertation, 1918.

PLATE III



WATER-VAPOR BANDS

λ 3064 and λ 2811 bands. *A*, second order. *B*, first order.

follow roughly the demands of the quantum theory of band spectra. But because of some very definite deviations from the theory, which might conceivably be due to errors in the assignment of some of the lines into series, new experimental data was considered to be necessary. Since the λ2811 band appeared to have a similar structure, an analysis of this band to combine with and check that of the λ3064 band was desired. Therefore, photographs of the λ2811 band were taken with the method described above in the second order of a 21-foot concave grating, having a dispersion of approximately 1.3 A per mm. The wave-lengths were checked by measurements of three different plates. Iron lines were used as standards. Since Burns's tables of standard iron lines extend only[1] to λ2851, it was found necessary to use two *Fe* lines measured by Buisson and Fabry.[2] Since the exposures were necessarily long—from 6 to 8 hours—the comparison spectrum was put on both before and after the exposures. If any shift of the iron lines was noticeable, the plates were discarded. These wave-lengths can be considered accurate to 0.01 A. Reductions to vacuum frequencies were made by means of the tables of the Bureau of Standards.[3] Considerable care was taken in investigating all the possibilities of the assignment of these lines into series, with the results as given in Table I. The series notation is that of Heurlinger. Unresolved doublets are indicated by the letter *d* in the intensity column. The intensities are given on a scale of from 1 to 10, 10 indicating the strongest line.

These lists contain every line of measurable intensity on the plates. Because of the marked similarity of the structure of this band as plotted in Figure 2 to that of the λ3064 band, one must conclude that the two bands have a close relationship to each other. The application of the quantum theory of band spectra to these series verifies this conclusion.

This theory assumes that a band line is emitted by a molecule when there is a change in its electron configuration either with or without a simultaneous change in its angular momentum. By the

[1] *Scientific Papers, U.S. Bureau of Standards,* No. 251, 1915

[2] *Astrophysical Journal,* **28,** 169, 1908.

[3] *Scientific Papers,* No. 327, 1918.

TABLE I

DETAILS OF THE λ 2811 BAND

m	Int.	λ(Air)	ν(Vacuum)	*m*	Int.	λ(Air)	ν(Vacuum)
		$R_1^1(m)$				$R_2^1(m)$	
2	0	2824.080	35399.37	2	3	2813.998	35526.18
3	2	2821.284	35434.44	3	4	2812.990	35538.91
4	9*d*	2819.127	35461.55	4	4	2812.158	35549.42
5	4	2817.552	35481.37	5	4	2811.573	35556.81
6	4	2816.505	35494.56	6	5*d*	2811.330	35559.89
7	8*d*	2815.983	35501.15	7	5*d*	2811.330	35559.89
8	8*d*	2815.983	35501.15	8	4	2811.770	35554.32
9	4	2816.410	35495.75	9	4	2812.568	35544.24
10	8*d*	2817.321	35484.28	10	4	2813.759	35529.20
11	4	2818.653	35467.52	11	4	2815.341	35509.24
12	4	2820.483	35444.50	12	8*d*	2817.321	35484.28
13	6*d*	2822.729	35416.30	13	4	2819.797	35453.13
14	3	2825.532	35381.17	14	6*d*	2822.729	35416.30
15	3	2828.759	35340.80	15	3	2826.029	35374.95
16	2	2832.483	35294.34	16	3	2829.865	35327.00
17	2	2836.723	35241.59	17	10*d*	2834.142	35273.69
18	1	2841.453	35182.93	18	1	2838.966	35213.75
19	0	2846.727	35117.76	19	0	2844.295	35147.77
		$Q_1^1(m)$				$Q_2^1(m)$	
2	10*d*	2829.247	35334.71	2	9*d*	2819.127	35461.55
3	5	2828.955	35338.36	3	9	2820.658	35442.30
4	6	2829.353	35333.39	4	9	2822.394	35420.51
5	7	2830.313	35321.41	5	10	2824.375	35395.66
6	10*d*	2831.812	35302.71	6	10	2826.654	35367.13
7	10	2833.822	35277.67	7	10*d*	2829.247	35334.71
8	10	2836.296	35246.89	8	10	2832.200	35297.87
9	10*d*	2839.211	35210.72	9	10	2835.511	35256.66
10	10	2842.618	35168.52	10	10*d*	2839.211	35210.72
11	10	2846.447	35121.21	11	10	2843.275	35160.39
12	10	2850.717	35068.60	12	10	2847.755	35105.08
13	10	2855.431	35010.72	13	10	2852.646	35044.90
14	10	2860.602	34947.43	14	10	2857.965	34979.68
15	10	2866.227	34878.85	15	10	2863.711	34909.49
16	10	2872.317	34804.90	16	10	2869.904	34834.17
17	9	2878.872	34725.65	17	10	2876.553	34753.65
18	10*d*	2885.913	34640.94	18	9	2883.686	34667.69
19	10*d*	2893.472	34550.45	19	8	2891.292	34576.49
20	10*d*	2901.540	34454.38	20	7	2899.416	34479.61
21	9*d*	2910.049	34353.64	21	8*d*	2907.993	34377.94
22	7	2919.185	34246.14	22	1	2916.917	34272.76
23	0	2928.885	34132.72				

TABLE I—*Continued*

m	Int.	λ(Air)	ν(Vacuum)	m	Int.	λ(Air)	ν(Vacuum)
	$P_1^1(m)$				$P_2^1(m)$		
2.....	10*d*	2831.812	35302.71	2.....	5	2821.682	35429.44
3.....	10*d*	2834.142	35273.69	3.....	5	2825.783	35378.03
4.....	9	2837.097	35236.95	4.....	6	2830.073	35324.40
5.....	9	2840.648	35192.90	5.....	8	2834.606	35267.91
6.....	9	2844.719	35142.54	6.....	8	2839.435	35207.94
7.....	9	2849.284	35086.25	7.....	9	2844.569	35144.39
8.....	8	2854.302	35024.56	8.....	9	2850.039	35076.95
9.....	8	2859.761	34957.71	9.....	9	2855.845	35500.64
10.....	8	2865.654	34885.83	10.....	9	2862.024	34930.08
11.....	8	2871.978	34809.01	11.....	9	2868.571	34850.35
12.....	8	2878.723	34727.45	12.....	9*d*	2875.500	34766.38
13.....	8	2885.841	34641.81	13.....	9	2882.840	34677.86
14.....	10*d*	2893.472	34550.45	14.....	10*d*	2890.522	34585.70
15.....	10*d*	2901.540	34454.38	15.....	8	2898.708	34488.03
16.....	9*d*	2910.049	34353.64	16.....	8	2907.261	34386.59
17.....	8	2918.902	34249.46	17.....	8	2916.229	34280.84
18.....	8	2928.277	34139.80	18.....	8	2925.647	34170.49
19.....	7	2938.111	34025.54	19.....	8	2935.537	34055.38
20.....	6	2948.444	33906.30	20.....	6	2945.888	33935.72
21.....	4	2959.237	33782.65	21.....	4	2956.724	33811.36
22.....	0	2970.548	33654.02	22.....	0	2968.079	33682.02
	$R_1^2(m)$				$R_2^2(m)$		
4.....	0	2883.071	34675.08	4.....	5*d*	2876.336	34756.27
5.....	4	2881.620	34692.55	5.....	7*d*	2875.500	34766.38
6.....	5*d*	2880.816	34702.23	6.....	6	2875.276	34769.08
7.....	0	2880.565	34705.25	7.....	7*d*	2875.500	34766.38
8.....	5*d*	2880.816	34702.23	8.....	5*d*	2876.336	34756.27
	$Q_1^1(m)$				$Q_2^2(m)$		
2.....	2	2892.890	34557.40	2.....	7	2882.308	34684.26
3.....	5	2892.694	34559.74	3.....	9	2883.950	34664.52
4.....	4	2893.183	34553.90	4.....	10*d*	2885.913	34640.94
5.....	8	2894.299	34540.57	5.....	8	2887.986	34616.07
6.....	8	2895.983	34520.49	6.....	10*d*	2890.522	34585.70
7.....	10*d*	2898.168	34494.46	7.....	8	2893.343	34552.00
8.....	9*d*	2900.935	34461.57	8.....	8	2896.577	34513.41
9.....	8*d*	2904.254	34422.18	9.....	6	2900.231	34469.93
10.....	8*d*	2907.993	34377.94	10.....	8*d*	2904.254	34422.18
11.....	7	2912.196	34328.31	11.....	7*d*	2908.503	34371.90

TABLE I—*Continued*

m	Int.	λ(Air)	ν(Vacuum)	m	Int.	λ(Air)	ν(Vacuum)
	$P_1^2(m)$				$P_2^2(m)$		
2.....	0	2895.453	34526.81	2.....	2	2884.841	34653.81
3.....	5	2897.834	34498.44	3.....	5	2889.021	34603.68
4.....	9*d*	2900.935	34461.57	4.....	10*d*	2893.472	34550.45
5.....	7	2904.538	34418.82	5.....	10*d*	2898.168	34494.46
6.....	8	2908.789	34368.52	6.....	7	2903.146	34435.33
7.....	8	2913.512	34312.81	7.....	7*d*	2908.503	34371.90
8.....	6	2918.794	34250.72	8.....	7	2914.221	34304.47
9.....	6	2924.524	34183.61	9.....	7	2920.341	34232.57
10.....	5	2930.748	34111.03	10.....	6	2926.873	34156.19
11.....	1	2937.467	34033.00	11.....	1	2933.825	34075.25
12.....	0	2944.640	33950.10	....	....	....	....

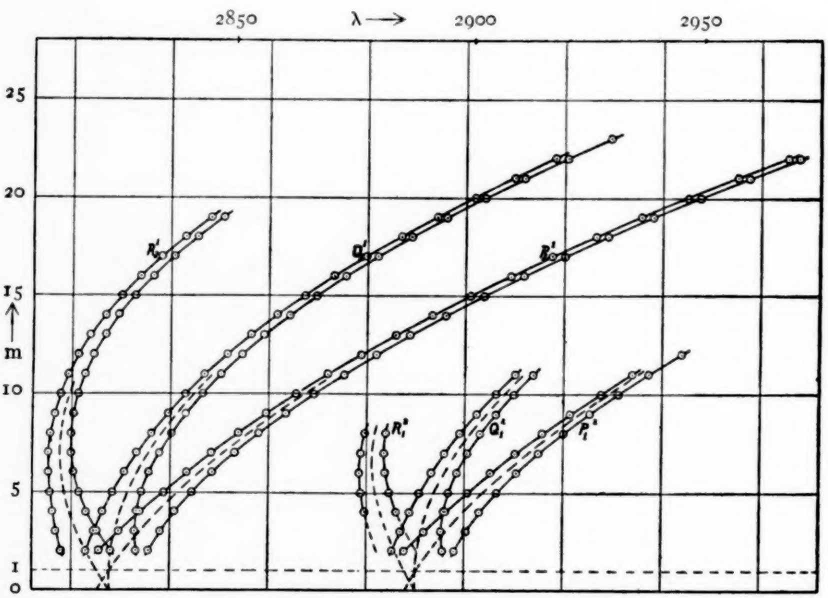


FIG. 2.—λ 2811 band

correspondence principle the rotational quantum m can only vary by the amounts $+1$ or -1, if at all. If we follow Curtis[1] by con-

[1] *Proceedings of the Royal Society*, A, **101**, 38, 1922.

sidering the jumps $m-1$ to m, m to $m-1$, and m to m in rotational quantum, the following three-line series result:

$$\left.\begin{aligned} P(m)&=\nu_0+h/8\pi^2I-(h/4\pi^2I)m+(h/8\pi^2)(1/I-1/I')m^2\\ R(m)&=\nu_0-h/8\pi^2I'+(h/4\pi^2I')m+(h/8\pi^2)(1/I-1/I')m^2\\ Q(m)&=\nu_0+(h/8\pi^2)(1/I-1/I')m^2 \end{aligned}\right\} \quad (1)$$

These formulae represent Heurlinger's P, R, and Q branches. The first term ν_0 is that part of the emitted frequency due solely to the electron rearrangement, while I and I' are respectively the moments of inertia of the molecule before and after this change.

Equations (1) hold for bands of the "whole-quantum" type, the distinguishing feature of which is the absence of two lines between the P and R branches at the ν_0 point. These two lines are to be identified with the transitions 1 to 0 and 0 to 1 in rotation quantum. In other words, the non-rotating molecule cannot exist. However, there are also bands in which only one missing line occurs at this point. These have been explained by Kratzer[1] by assuming "half-quantum" rotation states of the molecule. The rotation quantum m is put equal to $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, etc., instead of 0, 1, 2, etc.; and the missing line associated with the transition $\frac{1}{2}$ to $\frac{1}{2}$, this being a change in the sign of the rotation with no change in its numerical value.

A difficulty in the interpretation of our bands on the basis of this simple theory lies in the fact that the lines are all doublets. Now we might consider the series in separate sets, P_1, Q_1, R_1; P_2, Q_2, R_2, etc., with different values of ν_0 for each group; in other words, two different but closely allied electron changes occur in the molecule to produce them. It will be noticed that the P_1^1, R_1^1 and the P_2^1, R_2^1 branches intersect as they should at $m=\frac{1}{2}$, and are approximately reflections of one another about the line $m=\frac{1}{2}$. The Q branches, however, do not intersect the others at the $m=\frac{1}{2}$ point, and are not symmetrical about the $m=0$ line, as they should be according to the formulae.

On the other hand, if we place our branches in the middle of the doublets, as is done in the dotted curves of Figures 1 and 2, the result fits the theory somewhat better. The Q branches are

[1] *Sitz. d. Bayer. Akad.*, p. 107, March, 1922.

then symmetrical about the axis, while intersecting the P branches at $m=1$ and the R branches at $m=0$. Now Heurlinger[1] pointed out that in the λ 3064 band,

$$\left.\begin{aligned} P_2^p - P_1^p &= R_2^p - R_1^p - \text{const.} \\ P_2^p - P_1^p &= Q_2^p - Q_1^p + f(m), \end{aligned}\right\} \qquad (2)$$

relations which also hold true for the λ 2811 band. This fact would indicate that we are concerned here with true doublets. An explanation of the manner in which the band lines could be split up into doublets is offered in a recent paper by Kratzer.[2] The resultant moment of momentum of the electron motion in the molecule and the total moment of momentum are both quantized. Making the assumption that the two have the same direction, the mutual influence of the rotation and the electron motion is reckoned. Since the moment of momentum of the electron motion can have both + and − values, the energy term contains $m \pm \epsilon$ throughout, rather than just m as heretofore. The quantity ϵ is the quantum number for the electron motion. Kratzer shows that it has a value $\frac{1}{2}$ for the cyanogen bands, $\frac{1}{4}$ for some of the Hg bands, etc. In general ϵ can have any rational value.

The branches of these water-vapor bands, however, appear to be of the "whole-quantum" type, which are distinguished by the fact that there are two missing lines between the P and R branches at the ν_0 point. We therefore proceed to apply equation (1) to the mean values of the observed series of doublet frequencies to get an approximate idea of the moments of inertia. Were we to work with the values for the P_1, Q_1, R_1 and P_2, Q_2, R_2 series separately, the same values of I and I' would result.

Let the energy of the molecule before and after the deformation due to the change in the electron configuration be represented by $hF(m)$ and $hf(m)$ respectively. We can then represent a band of the simple type by the following frequency terms:

$$\begin{aligned} P(m) &= F(m-1) - f(m) \\ Q(m) &= F(m) - f(m) \\ R(m) &= F(m) - f(m-1)\ . \end{aligned}$$

[1] *Loc. cit.*

[2] *Annalen der Physik*, **71**, 72, 1923.

Obviously,

$$R(m)-Q(m)=Q(m-1)-P(m) , \tag{3}$$

and it is evident that this difference is a function only of the final state of the molecule. In fact, equations (1) show that

$$R(m)-Q(m)=(h/8\pi^2 I')(2m-1) .$$

If we take the difference

$$Q(m)-P(m)=F(m)-F(m-1) ,$$

it is evident that this is a function of the initial state only, and is equal to $(h/8\pi^2 I)(2m-1)$. To what extent this combination principle (3), which was announced by Heurlinger, holds true for the bands under discussion, is seen from Table II. The values of $h/8\pi^2 I'$ and $h/8\pi^2 I$, as given by the foregoing relations, have also been computed for all values of m.

TABLE II

m	1 $R(m)-Q(m)$	2 $Q(m-1)-P(m)$	3 $Q(m)-P(m)$	4 $h/8\pi^2 I$ from 3	5 $h/8\pi^2 I'$ from 1	6 $h/8\pi^2 I'$ from 2
	2811 Band					
2	64.7	....	31.9	(10.6)	21.6	....
3	96.3	72.1	64.5	12.9	19.3	14.4
4	128.1	109.7	96.7	13.8	18.3	15.7
5	160.6	147.0	128.1	14.2	17.8	16.3
6	192.3	183.2	159.7	14.5	17.5	16.7
7	224.5	219.6	190.7	14.7	17.3	16.9
8*	255.4	255.2	221.6	14.8	17.0	17.0
9	286.3	290.7	252.0	14.8	16.8	17.1
10	317.1	325.7	281.7	14.8	16.7	17.1
11	347.6	359.9	311.1	14.8	16.6	17.1
12	377.6	393.9	339.9	14.8	16.4	17.1
	3064 Band					
2	67.5	....	33.9	(11.3)	22.5	....
3	101.6	72.5	67.6	13.5	20.3	14.5
4	135.3	109.6	101.4	14.5	19.3	15.7
	.	.	.	.	.	.
	.	.	.	.	.	.
	.	.	.	.	.	.
12	397.5	393.8	358.1	15.6	17.3	17.1
13*	428.2	427.1	388.0	15.5	17.1	17.1
14	458.6	459.9	417.3	15.5	17.0	17.0
15	488.3	492.1	446.0	15.4	16.8	17.0

Some interesting facts are brought to light by these correspondence-principle relations. First, equation (3) is not satisfied except for one particular value of m—that value which marks the turning-point of the R branch. Column five shows that the final moment of inertia I', as reckoned from the $R(m)-Q(m)$ differences, is not a constant, whereas we see from column four that the $Q(m)-P(m)$ differences give very nearly the same value of the initial moment of inertia I for all values of m. It is interesting to note that equation (3) would be valid for all values of m if the R branch were rotated slightly about its turning-point in a counter-clockwise sense. In other words, *the R parabola has a tipped axis.* The final moment of inertia appears to get larger as the rotation quantum m increases. Perhaps the electron causing the deformation falls into orbits whose eccentricities depend upon the angular velocity of the molecule.

The so-called moment of inertia of a molecule, which consists of a group of particles all executing more or less intricate movements, is, then, an ambiguous and illusive quantity. But provided that its uncertain meaning is kept in mind, it is convenient for expressing certain relationships. With this reservation we will proceed to consider the moments of inertia of the carrier of these bands which are obtained from the theory. The following are the values of I, the initial, and I', the final moment of inertia, computed from Table II for the value of m, corresponding to the turning-point of the R branch in each case:

2811 Band	3064 Band
$I=1.87\times10^{-40}$ gr · cm²	$I=1.78\times10^{-40}$ gr · cm²
$I'=1.63\times10^{-40}$ gr · cm²	$I'=1.63\times10^{-40}$ gr · cm²

h has been taken as 6.55×10^{-27} erg · sec. and the wave-numbers changed to frequencies by multiplying by 3×10^{10}.

Can these results supply us with any clue as to the origin of these bands? Let us consider the common final moment of inertia of 1.63×10^{-40} gr. · cm.² If the OH ion is the carrier, this would mean that the nuclei were separated by a distance of 1.02 A. If the carrier is an approximately linear H_2O molecule, such as Kratzer and Sommerfeld[1] have pictured, rotating about the oxygen nucleus,

[1] A. Sommerfeld, *Atombau*, p. 539, 3d ed.

the distance between the two H nuclei would be 1.4 A. This gives 0.7×10^{-8} cm as the separation of the oxygen and adjacent hydrogen nuclei. Now one readily calculates from W. H. Bragg's[1] model of the crystal structure of ice that the distance between the O and H nuclei is 1.38×10^{-8} cm, while the separation of the H atoms is 2.26×10^{-8} cm. It is probable that the separations between the atoms of the H_2O molecules when in the gaseous state should be at least as great as these values. Therefore, this evidence appears to confirm our hypothesis that the OH ion is the carrier of these bands.

In this connection it might be mentioned that the theory of the width of spectral lines might give information as to the mass of the carriers of band spectra. This theory, as propounded by Lord Rayleigh,[2] connects the distribution of intensity in the line with the mass of the radiating center. This method might be used in some cases as a check on the results of the band spectra formulae, but of course could not be called upon to decide as to whether the H_2O molecule or the OH ion was the carrier of the bands under discussion.

The subsidiary bands which occur in both the λ 2811 and λ 3064 bands are similar in all respects to the main branches, but yield slightly different values for the quantities I and I' of the band formulae. Furthermore, the separation of the ν_0 points of the two subsidiary bands is 2790 cm^{-1}, while for the main bands it is 2980 cm^{-1}, a difference of 190 cm^{-1}. It is thus apparent that the carrying molecule for the subsidiary bands differs slightly in its structure from the molecule which radiates the main branches. Also these molecules are fewer in number than those of the main type, for the lines are all of less intensity. Now it is conceivable that when one of the hydrogen nuclei is removed from the H_2O molecule, it sometimes takes an electron with it, leaving but seven. In other cases the hydrogen nucleus alone is removed, leaving a complete shell of eight electrons in the OH ion. The main bands would thus be due, say, to OH with seven electrons, whereas the subsidiary bands would have their origin in OH ions with the full complement of eight electrons. Obviously the electrostatic repulsions and attrac-

[1] *Proc. Physical Soc. of London*, **34**, 98, 1922.

[2] *Phil. Mag.* (6), **29**, 274, 1915.

tions would be somewhat different in the two cases, thereby causing an alteration in the separation of the nuclei with a consequent change in the constants of the band formula.

There exists a set of fifteen lines which can be quite accurately represented by a parabolic formula, extending to the violet from the λ 3064 head. Now these lines cannot be assigned to the λ 2811 band, nor can they be associated in any way with the branches of the λ 3064 band. We can find no discussion of them in the literature, but yet they are found on every plate we have taken, and obviously belong to the same spectrum as the other series discussed above. Table III contains the wave-lengths and vacuum frequencies of these lines, the measurements being on the same second order plates as for the lines of the λ 2811 band.

TABLE III

Int.	λ(Air)	ν(Vacuum)	Int.	λ(Air)	ν(Vacuum)	Int.	λ(Air)	ν(Vacuum)
1*d*....	3021.282	33088.92	1...	3027.906	33016.53	5...	3044.328	32838.44
0.....	3021.751	33083.79	2...	3030.483	32988.47	6...	3048.565	32792.80
.....	3022.688	33073.54	3...	3033.416	32956.57	7...	3053.052	32744.61
.....	3024.002	33059.16	4...	3036.729	32920.61	8...	3057.730	32694.52
0.....	3025.746	33040.11	4...	3040.371	32881.18	8...	3062.520	32643.39

Apparently the λ 3021.282 line is a doublet, forming the head of the parabola, but it is impossible to pick up any higher members of the series. The λ 3062.5 line seems to end the progression, for there is no line in the λ 3064 band which could be the next member. Also, it is noteworthy that the lines of the series are singlets, this fact differentiating them from all the other branches considered. No explanation of the production of these lines presents itself at this time.

The author wishes to thank Dr. H. G. Gale and other members of the staff of Ryerson Laboratory for their assistance and encouragement. He also desires to take this opportunity to thank Mr. J. K. Morse, National Research Fellow, for his suggestions and criticisms on all phases of the investigation.

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THE ORBIT OF THE SPECTROSCOPIC BINARY 43 θ^2 ORIONIS

BY OTTO STRUVE

ABSTRACT

The period and the orbital elements of the spectroscopic binary 43 θ^2 Orionis were derived from the measurements of thirty-nine plates obtained with the Bruce spectrograph of the Yerkes Observatory. Within an interval of twenty years there seems to have been no appreciable change in the period, due to the resistance of the nebula. The velocities obtained from the emission lines of hydrogen and from the dark K line of *Ca* do not participate in the periodic shifts of the other lines. The mean velocity of these lines (emission and *Ca* K) is nearly the same as that of the nebula, and justifies the assumption that these lines are produced by the nebula. The velocity of the center of mass of the star is more than 20 km/sec. larger. The Orion nebula, as well as the calcium gases producing the stationary *Ca* line K, are nearly at rest with respect to the system of stars from which the motion of the solar system has been computed.

The orbit.—θ^2 Orionis ($\alpha = 5^h30^m.5$; $\delta = -5°29'$ [1900]; photographic mag., 4.95, spectral type, B1n) is one of the brighter stars located in the Orion nebula. Its Bond number is 685, and its position with respect to the trapezium is about 6^s following and 100″ south. The determination of the orbit of this star is especially interesting in connection with the question of a possible effect of the nebulous matter upon the period and upon other elements of the orbit.

The variable radial velocity of this star was announced by Messrs. Edwin B. Frost and Walter S. Adams, having been determined from four spectrograms taken in 1903 and 1904.[1] They found a range in velocity of 140 km/sec.

Further observations of θ^2 Orionis were made in 1919 by Dr. F. Henroteau at the Dominion Observatory in Ottawa.[2]

A short note by the writer on the bright lines of θ^2 Orionis was based on several observations made with the Bruce spectrograph of the Yerkes Observatory in 1922 and 1923.[3]

[1] *Astrophysical Journal*, **19**, 153, 1904.

[2] *Publications of the Dominion Observatory*, Ottawa, **5**, p. 19.

[3] *Astrophysical Journal*, **58**, 309, 1923.

The observations were continued at the Yerkes Observatory from October, 1923, until February, 1924, chiefly by Professor S. B. Barrett and the writer. In all, thirty-nine spectrograms could be measured for the determination of radial velocities.

The lines chiefly used for measurement, together with their wave-lengths and approximate widths, are collected in Table I.

TABLE I

Stellar Lines			Comparison Lines			
Element	λ	Width	Element	λ	Element	λ
Ca (K)......	3933.825	0.5	*Ti*........	3930.022	*Ti*........	4468.663
He..........	4026.352	2.2	*Fe*........	3969.413	*Ti*........	4481.435
*H*δ..........	4101.890	3.9	*Fe*........	4005.408	*Ti*........	4682.088
He..........	4143.928	2.2	*Ti*........	4028.497	*Ti*........	4856.203
*H*γ..........	4340.634	5.0	*Ti*........	4071.908	*Ti*........	4885.264
He..........	4388.100	3.2	*Ti*........	4132.235		
He..........	4471.676	3.2	*Fe*........	4144.038		
He..........	4685.898	3.1	*Ti*........	4338.084		
*H*β..........	4861.527	8.0	*Fe*........	4383.720		

It will be noticed that all stellar lines, with the exception of *Ca* K (λ 3934,), are very wide. Therefore, as was already pointed out by Professors Frost and Adams, the settings are difficult, and the internal agreement of the measurement is necessarily low. The hydrogen lines are complicated by the presence of narrow emission lines, and are therefore not suitable for accurate measurement. Although eight or nine stellar lines were measured on most of the spectrograms, it was found advisable to use for the orbit only the two helium lines λ 4472 and λ 4686. The other helium lines, as well as the hydrogen absorption lines, give substantially the same velocities, but their diffuse character makes an accurate measurement almost impossible. The *Ca* K line represents the typical case of a stationary calcium line, and the velocities derived from it had to be excluded from the orbit. They are discussed separately on page 165.

Table II contains the data of observation, together with the velocities resulting from the lines λ 4472 and λ 4686. Most of the plates were measured twice, some of them even three and four times. The weights were assumed proportional to the product of the number of times that a given plate was measured and one of the numbers,

1.0, 0.8, and 0.5, respectively, for plates designated as "good," "fair," or "poor." The first four plates were measured by E. B. Frost and W. S. Adams. The other plates were all measured by the writer. Hartmann-Cornu's dispersion formulae were computed for each spectrogram separately.

TABLE II

Plate	Date	G.M.T.	Observer	J.D. Red. to Epoch 2423700 +	Velocity	Weight
IB 240	1903, Dec. 31	$14^h 38^m$	F, S	56.644	+ 87.1	3.0
247	1904, Jan. 2	14 15	A, S	58.628	84.6	1.6
258	1904, Jan. 23	12 49	A, F, S	58.539	+ 42.5	3.2
273	1904, Jan. 29	13 13	F, S	43.527	− 48.2	2.0
517	1905, Feb. 13	14 30	F, S	46.058	+ 9.5	0.2
1302	1907, Dec. 30	16 58	Fox, S	44.711	− 21.9	0.5
1313	1908, Jan. 7	15 46	L, S	52.661	+ 79.4	0.8
1352	1908, Jan. 20	14 15	B, S	44.569	− 11.3	0.5
1405	1908, Feb. 3	14 34	B. S	58.582	+ 8.5	1.0
6423/1	1922, Feb. 20	13 47	B, S	58.473	81.7	0.8
6423/2	1922, Feb. 20	14 27	B, S	58.501	91.1	0.8
6788	1923, Mar 2	14 34	B, S	54.984	78.5	1.6
6794	1923, Mar. 7	15 23	B, S	60.108	28.5	2.0
6810	1923, Mar. 16	14 00	σ, S	47.931	25.4	1.0
6824	1923, Mar. 23	14 12	B, S	54.980	96.1	3.0
6831	1923, Mar. 26	14 02	B, S	57.933	+ 59.7	1.6
6840	1923, Mar. 30	13 51	B, S	61.925	− 29.9	1.0
IR 7146	1923, Oct. 2	22 39	B, S	38.002	+ 35.6	0.5
7161	1923, Oct. 7	21 33	B, S	42.956	− 60.4	1.6
7167	1923, Oct. 14	21 03	B, S	49.955	+ 61.6	1.6
7173	1923, Oct. 20	20 37	σ, S	55.917	100.1	1.0
7183	1923, Oct. 21	21 41	B, S	56.961	85.2	1.0
7191	1923, Oct. 22	20 58	σ, S	57.932	64.1	2.0
7201	1923, Nov. 6	19 33	σ, S	51.844	112.1	1.0
7210	1923, Nov. 10	20 04	B, σ, S	55.865	+119.0	1.6
7227	1923, Nov. 17	19 48	B, σ, S	41.825	− 82.6	2.0
7236	1923, Dec. 2	17.10	B, S	56.715	+111.2	1.6
7248	1923, Dec. 15	18 42	σ, S	48.750	51.0	2.0
7265	1924, Jan. 5	16 10	σ, S	48.616	70.6	2.0
7268	1924, Jan. 6	16 38	B, S	49.635	71.6	1.0
7274	1924, Jan. 12	14 23	σ, S	55.541	102.8	2.0
7276	1924, Jan. 12	17.50	σ, S	55.685	+105.6	2.0
7283	1924, Jan. 19	14 14	B, σ, S	41.506	− 74.8	1.0
7285	1924, Jan. 19	17.38	σ, S	41.648	64.1	1.0
7291	1924, Jan. 20	14 51	B, σ, S	42.532	− 92.6	1.6
7297	1924, Jan. 26	13 17	B, σ, S	48.466	+ 58.4	2.0
7299	1924, Jan. 26	16 08	σ, S	48.585	+ 73.4	1.0
7308	1924, Feb. 10	13 04	B, S	42.428	− 95.5	1.0
7310	1924, Feb. 10	16 14	B, S	42.560	− 84.4	1.0

In the column "Observer" the following designations are used: F = E. B. Frost; A = W. S. Adams; Fox = Philip Fox; L = O. J. Lee; B = S. B. Barrett; σ = Otto Struve; S = F. R. Sullivan.

On account of the inaccuracy of the individual measures, the plates were grouped and the following thirteen normal velocities were formed.

TABLE III

J.D.	Velocity	Weight	O.–C.*	O.–C.†
3741.660	− 76.0	0.5	+13.2	+ 0.1
3742.619	81.7	0.7	+ 0.2	− 3.6
3744.269	− 37.6	.4	− 7.0	− 4.6
3746.992	+ 20.1	0.2	17.5	− 8.5
3748.604	61.9	1.0	1.2	+ 2.0
3749.785	64.3	0.3	−12.8	−13.3
3752.252	97.6	.2	+ 2.9	− 3.4
3754.962	110.0	.6	10.5	+ 3.3
3755.752	107.2	.9	10.5	+ 3.8
3756.773	93.6	.8	+ 3.7	− 1.3
3757.933	62.2	0.5	−13.6	−15.7
3758.545	55.6	1.0	− 8.7	+ 6.2
3760.325	+ 12.7	0.5	+ 7.9	+ 6.3

* From the preliminary elements. † From the final elements.

After a number of attempts the period was found to be

$$P = 21.029 \text{ days.}$$

There seems to have been no appreciable change in the period within the past twenty years for which observations are available. During this interval of time the binary has completed 350 revolutions.

Preliminary elements were determined by the graphical method of Lehmann-Filhés. The residuals appeared to be rather large, and a least-squares solution was carried through with the use of Schlesinger's tables and formulae.[1] The results are collected below. Further refinements did not seem necessary on account of the unavoidable inaccuracy of the measurements.

TABLE IV

THE ELEMENTS

	Preliminary	Corrections	Final	Probable Errors
γ	+35.7 km/sec.	(+1.1)	+36.8 km/sec.	
τ	$21^{d}029$	Adopted	$21^{d}029$	
e	0.35	−0.08	0.27	±0.02
K	95.0 km/sec.	−1.3 km/sec.	93.7 km/sec.	±3.6 km/sec.
ω	$157^{\circ}2$	$-2^{\circ}5$	$154^{\circ}7$	$\pm 3^{\circ}2$
T	$-2423741^{d}300$	$+0^{d}062$	$3741^{d}362$	$\pm 0^{d}314$
$a \sin i$	27,000,000 km		27,000,000 km	

[1] *Publications of the Allegheny Observatory*, **1**, 33.

The probable error of one normal velocity of weight unity is ± 4.1 km/sec. The value of $[p\Delta^2]$ was reduced from 595.8 to 292.3. The velocity-curve below represents the corrected orbit.

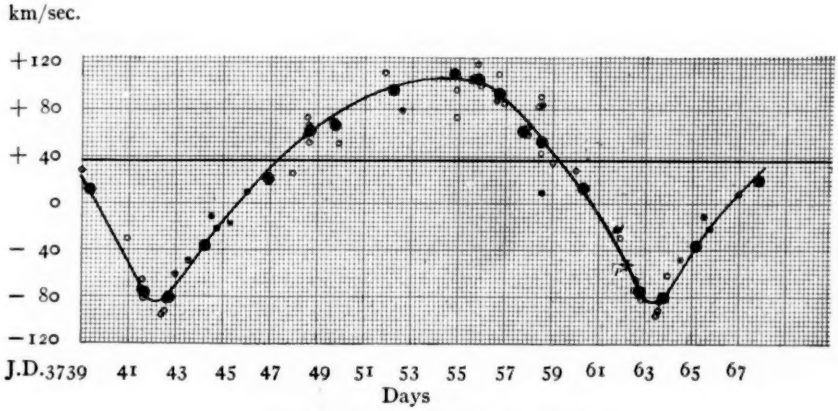

FIG. 1.—Velocity curve of 43 θ^2 Orionis

○ Velocities determined in 1922–1924
⊙ Velocities determined in 1903–1908
● Normal velocities

The value of K is decidedly greater than for an average binary having a period of twenty-one days. A glance at the curve on page 170 of this issue, giving the relation between P and K for 144 known spectroscopic binaries, shows that an average binary, having a period of twenty-one days, should have a value of K of the order of 45 km/sec. In the case of θ^2 Orionis, K is more than twice as large. This divergence cannot be explained as a result of inclination only, since in an average spectroscopic binary this inclination is such that $(\sin^3 i)$ av. $=0.65$. Therefore, even under the assumption that $i=90°$, K cannot amount to as much as 94 km/sec. It is also improbable that the mass ratio in this star is much larger than the average $(m_2/m_1=0.8)$. Only lines due to the stronger component of the binary (m_1) appear on our plates. This makes it probable that $m_2<m_1$ and, accordingly, the probability is small that $m_2/m_1>0.8$. The large value of K must evidently be attributed to a large total mass of the star, which may easily amount to as much as ten to twenty times the mass of the sun.

The bright lines.—On our early plates emission lines were found only in the case of Hβ. Later plates, especially those taken with the new camera lens of the Bruce spectrograph, designated by IR, show that emission lines appear also occasionally on Hγ, and in one or two instances also on Hδ. These emission lines are not always visible, and the variability of their intensity was pointed out last year.[1] It can now be shown that this variability is purely an effect of background. The bright lines—black on the plates—are visible only when they are superposed over the absorption lines due to the star. Since they do not participate in the periodic shifts of the absorption lines, but show a stationary velocity, they are sometimes superposed over the background of the continuous spectrum, on account of the large amplitude of the velocity variation of the absorption lines. When this happens, they are not visible, because the contrast is not sufficient to bring them out on a photograph. However, their presence is clearly demonstrated by the fact that they vitiate the velocities obtained from the hydrogen absorption lines, making the measured velocities greater near maximum and smaller near minimum. There seems to be no doubt that the bright lines are due to the Orion nebula. The mean velocity obtained from all plates on which bright Hβ and Hγ had been measured is $+17.6$ km/sec. A plate of the Orion nebula, taken near θ^2 Orionis, gave a velocity of $+16.5$ km/sec. This agreement is as good as can be expected. The nebular lines do not appear on our plates of θ^2 Orionis. From a comparison with the spectrum of θ^1 Orionis, where *Nu* λ 5007 is decidedly stronger than Hβ, we should expect to see the nebulium line in the spectrum of θ^2 Orionis, even though it is superposed on the continuous spectrum of the star. Its absence is apparently due to the weakening of the nebular lines in the outer regions of the nebula, as compared with the hydrogen lines. This effect was first noticed by W. W. Campbell[2] and later confirmed by C. Runge.[3]

Direct photographs of the Orion nebula show a small condensation around θ^2 Orionis, in which the star is placed somewhat eccentrically.

[1] *Astrophysical Journal*, **58**, 139, 1923. [2] *Ibid.*, **9**, 312, 1899.

[3] *Ibid.*, **8**, 32, 1898; *Astronomische Nachrichten*, **145**, 227, 1897.

The Ca line K.—The velocities obtained from the line λ 3934 were reduced with the same period as that found from the other lines, and plotted together with the velocities from the emission lines of hydrogen. There appears to be no connection between the period of 21.029 days and the velocities of either the line K or the emission lines. However, some of the plates give values for the calcium lines which deviate considerably from the average, and remeasurement did not remove these discrepancies. Table V contains the results of the measurements.

TABLE V

Plate	Vel.	Plate	Vel.	Plate	Vel.
6794	+15.2	7227	+24.7	7283	+25.5
6810	15.5	7248	8.9	7285	+13.4
6824	45.2	7265	15.2	7291	− 3.3
6831	10.3	7268	9.4	7297	− 1.9
6840	24.6	7274	0.7	7299	+ 1.6
7183	+ 5.3	7276	+10.2	7308	+ 7.7

The mean of these velocities is +12.7 km/sec., which is very close to the velocity of the Orion nebula in this region. The difference between these two velocities lies within the errors of measurement. Therefore, the writer is inclined to believe that the calcium atmosphere surrounding the star is the Orion nebula. Whether or not the variations illustrated in Table V are real cannot be decided from the material at hand. On most of our plates the region around λ 3934 is somewhat underexposed, and this may produce errors in the interpretation of the line. On the other hand, all plates indicated by the letters "IR" were taken with the new camera lens of the Bruce spectrograph (designed by Dr. F. E. Ross), which gives best definition near λ 3900, the definition being almost as good in the region of λ 4800 and only a little less sharp in the center of the plate, near λ 4500.

The velocity of the center of mass of the star is +37 km/sec., which is quite different from the velocities of the nebula (+16.5 km/sec.), the emission lines of hydrogen (+17.6 km/sec.), and the sharp calcium line (+12.7 km/sec.).

In connection with Dr. J. S. Plaskett's recent paper on stationary calcium clouds in space,[1] it may be stated that the component of the solar motion in the direction of the Orion nebula amounts to about 18 km/sec. Therefore, the various velocities corrected for the motion of the solar system would be as follows:

TABLE VI

θ^2 Orionis (center of mass)	+19 km/sec.
Hydrogen emission lines	± 0 km/sec.
Orion nebula	− 2 km/sec.
Calcium K (stationary)	− 5 km/sec.

YERKES OBSERVATORY
February 19, 1924

[1] *Monthly Notices of the Royal Astronomical Society*, **84**, 80, 1923.

ON THE NATURE OF SPECTROSCOPIC BINARIES OF SHORT PERIOD

By OTTO STRUVE

ABSTRACT

An investigation of observational data for 144 spectroscopic binaries shows: (1) that the relation between the period P and the semi-amplitude of velocity variation K is in agreement with the double-star hypothesis for periods longer than 2.4 days; (2) that for Cepheid variables there is no change in K corresponding to a change in P; (3) that spectroscopic binaries having periods less than 2.4 days consist of two distinct groups of stars. The more numerous group is characterized by small values of K, which is practically the same as for Cepheid variables. Apparently these stars are not double stars, but are closely related to the Cepheids. The other, less numerous group is characterized by large values of K, indicating that these stars are actually double. On account of their small orbital dimensions, these stars are probably to a large extent eclipsing variables, as, for example, W Ursae Majoris. In a given particular case it is not always possible to decide whether a star belongs to the "pseudo Cepheids" or to the real double stars. This is due to the uncertainty of i in any given case.

The average total mass was determined for those of the 144 spectroscopic binaries which are probably actual double stars. This mass was found to be about three times the mass of the sun. The average spectral type of the stars investigated was found to be A4.

The periods of all known spectroscopic binaries range from 2.5 hours, in the case of γ Ursae Minoris,[1] to some 50 years in the case of visual double stars for which the spectrographic method indicates variable radial velocities. It is important to know whether all stars showing periodic displacements of spectral lines are of the same physical nature. Various investigators have pointed out that the physical nature of the Cepheid variables is probably different from that of the ordinary double stars. One of the most outstanding characteristics of the Cepheids is the small value of the semi-amplitude of the velocity variation K. The same fact applies also to almost all spectroscopic binaries having very short periods, such as β Cephei, τ Cygni, γ Ursae Minoris.

Theoretically the value of K depends on the period P in the following way (the numbers in brackets are logarithms):[2]

$$(m_1+m_2)\sin^3 i=[3.01642-10](K_1+K_2)^3P(1-e^2)^{\frac{3}{2}} \tag{1}$$

[1] *Popular Astronomy*, **31**, 90, 1923.

[2] R. G. Aitken, *The Binary Stars*, p. 203.

where

m_1 = the mass of the primary component
m_2 = the mass of the secondary component
i = the angle of inclination to the tangent plane
K_1 = the semi-amplitude of the velocity variation for m_1
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Since

$$\frac{m_1}{m_2}=\frac{K_2}{K_1}$$

this equation may be written:

$$(m_1+m_2)\sin^3 i=[3.91642-10]\left(K_1+\frac{m_1}{m_2}K_1\right)^3 P(1-e^2)^{\frac{3}{2}}. \tag{2}$$

If we substitute in place of (m_1+m_2), $\sin^3 i$, $\frac{m_1}{m_2}$, and e their mean values, we may write, simply:

$$K=CP^{-\frac{1}{3}}, \tag{3}$$

C being a constant depending on the average mass of the stars under consideration. This formula represents, of course, only an average case and does not hold for any particular star. However, it is known that in double stars the dispersion in mass and in mass ratio is not very large, and therefore we are justified in assuming that for any sufficiently large group of double stars both the total mass and the mass ratio are constants.

In order to test the actual relation between K and P in spectroscopic binaries, the table on page 169 was made on the basis of 144 known orbits contained chiefly in R. G. Aitken's *The Binary Stars* in Table II, page 296.

The average spectral type for all stars is approximately A4.

The table shows definitely that for the mean of all stars K increases continuously from 4 km/sec. to 87 km/sec., when P diminishes from 9 years to 2.45 days. The individual values have, of course, a large dispersion, due to different orbital inclinations, perhaps also due to actual differences in mass and mass ratio.

On the other hand, it is evident that no relation exists between P and K for Cepheid variables; K remains practically constant for Cepheids with periods ranging between 0.2 day and 17 days. The mean value of K for all 15 Cepheids included in the table is 16.0 km/sec.

TABLE I

No. of Group	No. of Stars	P	No. of Cepheids	K (for all Stars)	K (except Cepheids)	K (Cepheids alone)	Average Spectral Type
				km/sec.	km/sec.	km/sec.	
1.......	6	$0^{d}.28$	3	17.5	17.5	18.3	B6
2.......	10	1.45	....	59.1	59.1	....	A3
3.......	12	2.43	....	87.3	87.3	....	B9
4.......	23	3.88	4	63.2	73.5	14.1	A2
5.......	26	7.0	5	47.3	54.2	18.8	A1
6.......	17	14.2	3	45.2	52.4	11.4	A6
7.......	17	36.0	....	45.5	45.5	....	A1
8.......	12	109.1	....	30.3	30.3	....	A0
9.......	8	218.2	....	28.7	28.7	....	F5
10.......	8	820.	....	11.7	11.7	....	F1
11.......	5	3199.	....	4.0	4.0	....	G3

For this reason it was found necessary to exclude the Cepheids from the various groups in Table I, and column (6) of this table contains the value of K for all stars except the Cepheids. The increase of K with decrease of P is still more convincing than in the case of all stars, including the Cepheids. However, this increase of K continues only down to $P=2.45$ days.

At $P \leq 2.45$ days, the curve combining period and semi-amplitude K falls rapidly. This decline is contradictory to equation (3), and can be explained only under the assumption that groups (1) and (2) of Table I consist not only of ordinary double stars, but also of stars having variable radial velocities which are showing not real double stars.

The results obtained from Table I are shown graphically in Figure 1 (p. 170).

Reference should here be made to a certain class of eclipsing binaries for which the existence of two separate components revolving around one another is well established. Some of these stars, for example, W Ursae Majoris, have periods decidedly shorter than one day: $0^{d}.334$ for W Ursae Majoris, $0^{d}.321$ for

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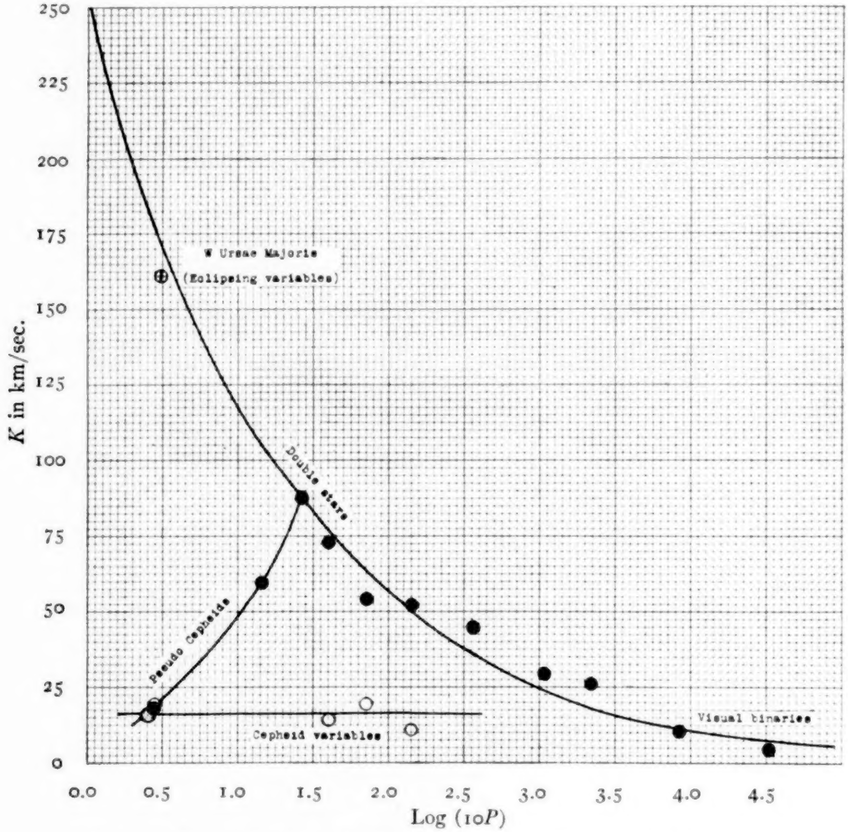

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Group	log C	Group	log C
3	2.0695	9	2.2375
4	2.0425	10	2.0395
5	2.0257	11	1.7704
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7	2.1768	Mean	2.0695
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Using this value of log C an ephemeris was computed from formula (3) as shown in Table III. It will be seen from Table III that

TABLE III

P	K (computed)	K (observed)	log (10P)
	km/sec.	km/sec.	
$0^d.10$	252.8	....	0.00
0.30	175.3	(161)*	0.48
1.00	117.4	....	1.00
1.50	102.5	....	1.18
2.45	87.3	87.3	1.38
3.88	78.2	73.5	1.59
7.0	61.3	54.2	1.85
14.2	48.5	52.4	2.15
36.0	35.5	45.5	2.56
109.1	24.5	30.3	3.04
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W Ursae Majoris falls near the upper branch of the curve. This is in good agreement with the double-star hypothesis. All other stars having periods less than a day have a mean value of K which is about 10 times smaller than that required by the curve.

Table IV contains a list of 10 stars, for which the values of P and K are known with considerable accuracy. This table is copied from a paper by the present writer, entitled "A Study of Spectroscopic Binaries of Short Period," which is now ready for publication.[1]

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Star	P	K
		km/sec.
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γ Lyrae	.15	12.5 (estimated)
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57 Ursae Majoris	.500	15.0 (estimated)
RR Lyrae	0.5668	22.2
Mean for 10 stars	0.27	15.1

It is an interesting fact that the mean value of K for 10 spectroscopic binaries having periods less than one day ($K = 15.1$ km/sec.) almost exactly coincides with the mean value of K for 15 Cepheid variables included in Table I ($K = 16.0$ km/sec.). This seems to be a strong argument in favor of the hypothesis that most spectroscopic binaries having periods shorter than one day have a marked analogy to the Cepheids. Such a supposition is strengthened by the fact that a considerable number of these stars having short periods are known, chiefly from photo-electric measures, to vary in brightness with amplitudes of a few hundredths of a magnitude.

It is highly probable that the decline of our curve for periods shorter than 2.45 days is to be explained as a result of superposition of two frequency-curves of stars having variable radial velocities.

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The one frequency-curve is that of the ordinary double stars, following the law expressed in equation (3), which is nothing else than Kepler's third law. These stars are real spectroscopic "binaries." Their number undoubtedly decreases with a decrease in period, since for periods shorter than 2.45 days only dense dwarfs can exist as completely separated double stars.

The other frequency-curve represents such stars as β Cephei and γ Ursae Minoris, which we may call provisionally "pseudo-Cepheids." We do not know at which value of P this curve reaches a maximum. But it is evident that this curve must be such that, for group (1) in Table I, the percentage of "pseudo-Cepheids" is much greater than for group (2).

In any given case it is not always possible to distinguish between the two groups. This is especially true for stars having periods longer than 2.45 days. If the value of K is large, we shall doubtless recognize the star as a real double star, but if K is small, this may be due either to a small value of the inclination or to the fact that the star is a "pseudo-Cepheid."

The abnormal character of the stars designated as "pseudo-Cepheids" may be explained in three different ways.

1. The total mass of these systems may be much smaller than that of all single stars or binaries of longer period. This alternative seems quite improbable in view of the fact that the spectral characteristics of these stars do not exhibit any such anomalies as could be expected in the case of small stellar masses.

2. The mass ratio may be very far from one, the system representing some sort of transition between a regular double star and a single bright star followed by a planet. This explanation does not seem probable, since we do not know of any such systems among binaries with longer periods. Besides, the sudden changes in the velocity-curves, which apparently occur in all of these stars, contradict any explanation by simple orbital motion.

3. These stars are perhaps not double stars at all and the observed line-shifts may be due to some other cause than orbital motion. This alternative must be regarded as the most probable at the present time. It has been shown in the case of Cepheid variables that periodic line-shifts can be attributed to pulsations.

It is not impossible that such pulsations produce also the periodic line-shifts in "pseudo-Cepheids."

Equation (2) enables us to draw some conclusions as to the average mass of those spectroscopic binaries which are actual double stars and which are included in groups (3) to (11) of Table I. The mean value for m_2/m_1 is, according to Aitken,[1] approximately 0.8; the mean value of $\sin^3 i$ is about[2] 0.65; e^2 may be neglected. If such substitutions are made, we find with the aid of the constant C of our curve, for the average spectral type A4, a total mass:

$$m_1+m_2=3\,\odot$$

This value is, however, not characteristic of A-type stars in general. A spectroscopic binary is more easily discovered if K is large, which happens if i is near 90° and if the total mass of the system is large. The effect of selection in i has been compensated for in assuming the mean value of $\sin^3 i$ to be 0.65 and not 0.59, which results from the formula:

$$(\sin^3 i)_{\text{average}}=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin^4 i\,di\,d\phi=\tfrac{3}{16}\pi=0.59.$$

The effect of selection in mass cannot be compensated for so easily. Therefore, the value of (m_1+m_2) found above is probably somewhat in excess of the average mass of all A-type stars.

A spectroscopic binary, having a very short period, has necessarily also a small value of a, the semi-major axis of its orbit. For this reason many of the stars which fall on the upper branch of our curve are known to be eclipsing variables. Table I includes 14 stars having periods shorter than 3.0 days, which are beyond doubt actual double stars. Of these, 8 stars are known to be eclipsing binaries. It is not impossible that some of the remaining stars will also be found to show eclipsing effects.

YERKES OBSERVATORY
January 29, 1924

[1] *The Binary Stars*, pp. 205, 207.

[2] W. W. Campbell, *A Second Catalogue of Spectroscopic Binaries*, *Lick Observatory Bulletins*, **6**, 39, 1910; F. Schlesinger and R. H. Baker, *Publications of the Allegheny Observatory*, **1**, 146.

MEAN PARALLAXES OF STARS OF SMALL PROPER MOTION[1]

By FREDERICK H. SEARES

ABSTRACT

Distribution functions for small values of stellar velocity.—In *Contribution* No. 272 it was shown that if the logarithms of space-velocities have a Gaussian distribution the frequencies of small tangential velocities are sensibly proportional to the velocity. Here it is found that for small values of the space-velocity the Gaussian law for space-velocity cannot hold, or at least is very improbable. This, however, does not appreciably affect the results for tangential velocity. General considerations show that the *frequency function for small tangential velocities*, whether corrected or uncorrected for solar motion, has the form

$$N(T)dT=(D_0+D_1T+D_2T^2+\ .\ .\ .\ .\ )TdT \qquad (T \lesssim \tfrac{1}{3}\underline{T})$$

where D_1, D_2, etc., are zero or small in comparison to D_0. For $T>\underline{T}/3$ the Gaussian function of $\log T$ is applicable, as shown in *Contribution* No. 272.

Mean parallaxes for small proper motions.—The law for small tangential velocities, combined with the luminosity and density functions, leads to simple algebraic expressions, equations (27) and (30), for the mean parallaxes of stars of magnitude m and proper motion zero. The *mean parallaxes for* $\mu=0$ thus found are finite, whereas a formula of the type proposed by Kapteyn, which presupposes a Gaussian distribution of $\log T$ for all values of T, gives zero values of the parallax for $\mu=0$. For *small* but finite *proper motions*, values of T both larger and smaller than $\underline{T}/3$ must be taken into account, and both forms of the frequency function appear in the expression for mean parallax. The resulting formulae, equations (38) and (47), are functions of the probability integral. Computations for $m=5$ and $\mu=0''.005$, $0''.01$, $0''.02$, and $0''.04$ show that the deviations from Kapteyn's formula begin to be appreciable at $\mu=0''.03$ or $0''.04$. The limiting μ for other values of m should not differ greatly from this value.

Comparisons with van Rhijn's mean parallaxes.—The formulae developed here for zero and small proper motions give mean parallaxes agreeing with van Rhijn's tables as well as can be expected in view of our imperfect knowledge of the coefficients of the density function.

Kapteyn's mean parallax formula

$$\log \bar{\pi}=A+Bm+C\log\mu \qquad (1)$$

gives zero values of the parallax for stars having zero proper motions. But, as pointed out by Strömberg,[2] the observational evidence indicates clearly that with decreasing μ the mean parallax for a given m converges toward a finite limit. To take account of this circumstance Strömberg proposed the substitution of $C\log(\mu+c)$ for the last term in (1), where, for a given magnitude, c is a constant.

[1] *Contributions from the Mount Wilson Observatory*, No. 282.

[2] *Mt. Wilson Contr.*, No. 144; *Astrophysical Journal*, **47**, 7, 1918; *Mt. Wilson Contr.*, No. 170, 1919.

where

m_1 = the mass of the primary component

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i = the angle of inclination to the tangent plane

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e = the eccentricity of the orbit

Since

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this equation may be written:

$$(m_1+m_2)\sin^3 i=[3.91642-10]\left(K_1+\frac{m_1}{m_2}K_1\right)^3 P(1-e^2)^{\frac{3}{2}}. \qquad (2)$$

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For this reason it was found necessary to exclude the Cepheids from the various groups in Table I, and column (6) of this table contains the value of K for all stars except the Cepheids. The increase of K with decrease of P is still more convincing than in the case of all stars, including the Cepheids. However, this increase of K continues only down to $P = 2.45$ days.

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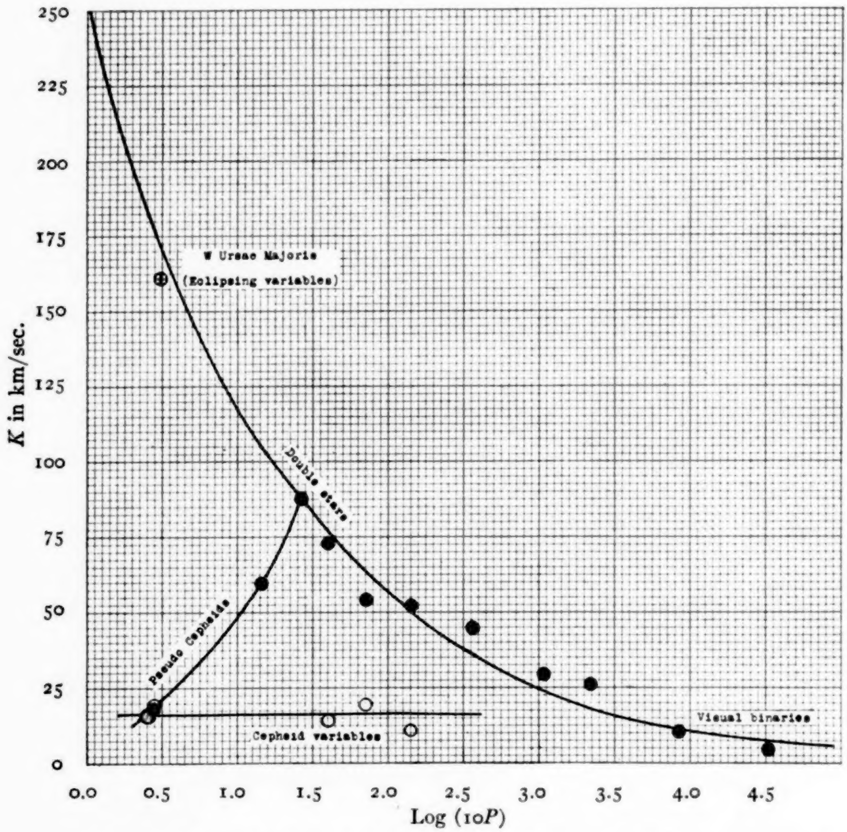

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3. These stars are perhaps not double stars at all and the observed line-shifts may be due to some other cause than orbital motion. This alternative must be regarded as the most probable at the present time. It has been shown in the case of Cepheid variables that periodic line-shifts can be attributed to pulsations.

It is not impossible that such pulsations produce also the periodic line-shifts in "pseudo-Cepheids."

Equation (2) enables us to draw some conclusions as to the average mass of those spectroscopic binaries which are actual double stars and which are included in groups (3) to (11) of Table I. The mean value for m_2/m_1 is, according to Aitken,[1] approximately 0.8; the mean value of $\sin^3 i$ is about[2] 0.65; e^2 may be neglected. If such substitutions are made, we find with the aid of the constant C of our curve, for the average spectral type A4, a total mass:

$$m_1+m_2=3\ \odot$$

This value is, however, not characteristic of A-type stars in general. A spectroscopic binary is more easily discovered if K is large, which happens if i is near 90° and if the total mass of the system is large. The effect of selection in i has been compensated for in assuming the mean value of $\sin^3 i$ to be 0.65 and not 0.59, which results from the formula:

$$(\sin^3 i)_{\text{average}}=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin^4 i\ di\ d\phi=\tfrac{3}{16}\pi=0.59.$$

The effect of selection in mass cannot be compensated for so easily. Therefore, the value of (m_1+m_2) found above is probably somewhat in excess of the average mass of all A-type stars.

A spectroscopic binary, having a very short period, has necessarily also a small value of a, the semi-major axis of its orbit. For this reason many of the stars which fall on the upper branch of our curve are known to be eclipsing variables. Table I includes 14 stars having periods shorter than 3.0 days, which are beyond doubt actual double stars. Of these, 8 stars are known to be eclipsing binaries. It is not impossible that some of the remaining stars will also be found to show eclipsing effects.

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[1] *The Binary Stars*, pp. 205, 207.

[2] W. W. Campbell, *A Second Catalogue of Spectroscopic Binaries, Lick Observatory Bulletins*, **6**, 39, 1910; F. Schlesinger and R. H. Baker, *Publications of the Allegheny Observatory*, **1**, 146.

MEAN PARALLAXES OF STARS OF SMALL PROPER MOTION[1]

By FREDERICK H. SEARES

ABSTRACT

Distribution functions for small values of stellar velocity.—In *Contribution* No. 272 it was shown that if the logarithms of space-velocities have a Gaussian distribution the frequencies of small tangential velocities are sensibly proportional to the velocity. Here it is found that for small values of the space-velocity the Gaussian law for space-velocity cannot hold, or at least is very improbable. This, however, does not appreciably affect the results for tangential velocity. General considerations show that the *frequency function for small tangential velocities*, whether corrected or uncorrected for solar motion, has the form

$$N(T)dT=(D_0+D_1T+D_2T^2+\ .\ .\ .\ .\ )TdT \qquad (T \lesssim \tfrac{1}{3}\underline{T})$$

where D_1, D_2, etc., are zero or small in comparison to D_0. For $T>\underline{T}/3$ the Gaussian function of $\log T$ is applicable, as shown in *Contribution* No. 272.

Mean parallaxes for small proper motions.—The law for small tangential velocities, combined with the luminosity and density functions, leads to simple algebraic expressions, equations (27) and (30), for the mean parallaxes of stars of magnitude m and proper motion zero. The *mean parallaxes for* $\mu=0$ thus found are finite, whereas a formula of the type proposed by Kapteyn, which presupposes a Gaussian distribution of $\log T$ for all values of T, gives zero values of the parallax for $\mu=0$. For *small* but finite *proper motions*, values of T both larger and smaller than $\underline{T}/3$ must be taken into account, and both forms of the frequency function appear in the expression for mean parallax. The resulting formulae, equations (38) and (47), are functions of the probability integral. Computations for $m=5$ and $\mu=0\rlap{.}''005$, $0\rlap{.}''01$, $0\rlap{.}''02$, and $0\rlap{.}''04$ show that the deviations from Kapteyn's formula begin to be appreciable at $\mu=0\rlap{.}''03$ or $0\rlap{.}''04$. The limiting μ for other values of m should not differ greatly from this value.

Comparisons with van Rhijn's mean parallaxes.—The formulae developed here for zero and small proper motions give mean parallaxes agreeing with van Rhijn's tables as well as can be expected in view of our imperfect knowledge of the coefficients of the density function.

Kapteyn's mean parallax formula

$$\log \bar{\pi}=A+Bm+C\log\mu \tag{1}$$

gives zero values of the parallax for stars having zero proper motions. But, as pointed out by Strömberg,[2] the observational evidence indicates clearly that with decreasing μ the mean parallax for a given m converges toward a finite limit. To take account of this circumstance Strömberg proposed the substitution of $C\log(\mu+c)$ for the last term in (1), where, for a given magnitude, c is a constant.

[1] *Contributions from the Mount Wilson Observatory*, No. 282.

[2] *Mt. Wilson Contr.*, No. 144; *Astrophysical Journal*, **47**, 7, 1918; *Mt. Wilson Contr.*, No. 170, 1919.

Van Rhijn, in his discussion of mean parallaxes in *Groningen Publication* No. 34, has also recognized the objection to equation (1), and has met the difficulty by another modification of Kapteyn's formula.

Equation (1), however, is not merely an empirical formula which conceivably might be in error when applied to values requiring an extrapolation; as first shown by Schwarzschild, it is also the analytical consequence of certain assumptions for the distribution of luminosity, density, and stellar velocity; and it has been shown in *Contributions* Nos. 272 and 273 that, with certain limitations, these assumptions are well justified. As a further definition of limitations, it is important to know the origin and the consequences of the contradiction between equation (1) and the observational result that the mean parallax approaches a finite limit as the proper motions become smaller and smaller.

General considerations show that the difficulty must lie in the frequency function for tangential velocity. Schwarzschild's assumption was that the logarithms of these velocities have a Gaussian distribution. The discussion in *Contribution* No. 272, which was based on the Gaussian distribution of the logarithms of the space-velocities found by Adams, Strömberg, and Joy,[1] shows that Schwarzschild's assumption is applicable for tangential velocities greater than about 10 km/sec., while the smaller velocities are well represented by

$$P(T_0)dT_0 \propto T_0 dT_0 \,. \tag{2}$$

Since the velocities of more than 90 per cent of the stars in a given volume of space are above the critical limit of 10 km/sec., the Gaussian function for $\log T_0$ can safely be used for the study of most statistical questions. In fact, it is generally applicable unless the proper motions considered are small. The condition to be satisfied can be written

$$T_0 = 4.74\,\mu\rho \gtrless 10 \text{ km/sec.} \tag{3}$$

in which the distance ρ is expressed in parsecs. For moderate and large values of μ, T_0 can fall below the limit only for stars very near

[1] *Mt. Wilson Contr.*, No. 210; *Astrophysical Journal*, 54, 9, 1921.

the sun, and these are usually a negligible fraction of the total involved in any general question of distribution.

Equation (3) shows that the value of the mean parallaxes for $\mu=0$ depends upon the behavior of the small tangential velocities; but it must be noted that (2) cannot be used without further inquiry to represent analytically the distribution of these velocities. Numerically, the frequencies near $T_0=0$ are small quantities, and there is consequently much uncertainty in deriving an analytical expression from the frequencies themselves. Moreover, (2) depends upon the validity of the adopted distribution function for space-velocity.

Adams, Strömberg, and Joy[1] found that the Gaussian function of log v:

$$F(\log v)\, d\, (\log v)=\frac{h_v}{\sqrt{\pi}}e^{-h_v^2(\log v-\log \underline{v})^2}d(\log v) \tag{4}$$

gives an excellent representation of the observed data; but it cannot be said that this expression is established for small values of v. The probability of a small v is inherently small, and the number of velocities near $v=0$ thus far observed is insufficient to define the distribution without ambiguity.

Rather simple general considerations lead, however, to the following conclusions: (1) Equation (4) is inapplicable to values of v less than about one-third the mean space-velocity. (2) The logarithms of small tangential velocities, whether corrected or uncorrected for solar motion, cannot possibly have a Gaussian distribution. (3) Tangential velocity, both corrected and uncorrected, can be expressed in the form

$$N(T)dT=(D_0+D_1T+D_2T^2\ .\ .\ .\ .)TdT\ , \tag{5}$$

where, for small values of T, the first term alone is of significance. With this established, it is easy to derive an expression for the mean parallax of stars of zero proper motion, which, formally at least, is in agreement with the facts of observation.

1. Consider the velocity vectors drawn from the origin $v=0$ for all the stars in a given volume of space, and, more particularly, the space-density of the terminals of these vectors. Without

[1] *Loc. cit.*

essential limitation of the argument, we may assume spherical symmetry about the origin, and, if $D(v)$ represent the space-density of the vector terminals, the true distribution function for v will then have the form

$$P(v)dv=4\pi D(v)v^2dv\ , \tag{6}$$

where $4\pi v^2dv$ is the volume element in the vector space.

If now it be supposed that the frequencies of the space-velocities can be represented by the Gaussian function (4), we shall have $P(v)=F(\log v)\text{Mod.}/v$, so that the expression for $D(v)$ becomes

$$D(v)=\frac{F(\log v)\text{Mod.}}{4\pi v^3}\ . \tag{7}$$

Since

$$v^3=e^{3\log v/\text{Mod.}}\ ,$$

it appears that $D(v)d(\log v)$ will be a Gaussian function of $\log v$, with a maximum at

$$\log v_m=\log \underline{v}-\frac{3}{2h_v^2\ \text{Mod.}}\ . \tag{8}$$

The value $h_v=2.62$ was found in *Contribution* No. 272. Hence $v_m=\underline{v}/3$, approximately, and for the typical value $\log \underline{v}=1.50$, $v_m=10$ km/sec. When the frequencies are referred to the interval dv instead of $d(\log v)$, the maximum occurs for a somewhat larger value of v; but in either case there is a sudden drop in the density of the vector terminals for small velocities and a zero value for $v=0$. This, to say the least, is very improbable. It is to be expected, therefore, that (4) will not represent the distribution of space-velocities for values less than $\underline{v}/3$.

2. Consider now the projection of the vector system on the tangential plane. The projected vectors represent the distribution of tangential velocity at the solar apices, or of tangential velocity generally, if freed from solar motion. For spherical symmetry of the space-vectors, we shall have circular symmetry in the plane about the origin $T=0$, and can write

$$N(T)dT=2\pi D_1(T)TdT\ , \tag{9}$$

where $N(T)dT$ is the true distribution function for corrected tangential velocity, $D_1(T)$ the surface density of the projected

vector terminals at the distance T from the origin, and $2\pi T dT$ the ring element of surface. It is evident from geometrical considerations that $D_1(T)$ cannot be zero for $T=0$. Even if $D(v)$ were zero at the origin, the projection of the space-vectors on the tangential plane would necessarily give a finite surface density of the terminals at $T=0$.

It follows at once that the distribution of small tangential velocities cannot be represented by a Gaussian function, say $G(\log T)d(\log T)$, for $D_1(T)$ would then be determined by

$$D_1(T)=\frac{G(\log T)\ \text{Mod.}}{2\pi T^2} \tag{10}$$

which is itself a Gaussian function of $\log T$, and hence zero at $T=0$. Further, the maximum of this function is at

$$\log T_1=\log \underline{T}-\frac{1}{h_T^2\ \text{Mod.}}\ . \tag{11}$$

For the typical case discussed in *Contribution* No. 272, equation (46) gives $\log \underline{T}=1.40$, and $h_T=2.35$; hence T_{max} is just under 10 km/sec. It is to be expected, therefore, that the Gaussian function of $\log T$ will fail for values of T less than this limit, and this, in fact, is just what was found in *Contribution* No. 272.

It is further evident from geometrical considerations that $D_1(T)$ is not only finite for $T=0$, but also continuous and free from rapid variations near the origin. Hence, for small values of T, $D_1(T)$ can be expressed in the form of a rapidly converging power series:

$$D_1(T)=a_0+a_1T+a_2T^2+\ \ldots\ldots \tag{12}$$

This expression, combined with (9), defines the distribution function for corrected tangential velocity. Its form is that of equation (5).

3. Tangential velocities uncorrected for solar motion (T_0) depend upon the distances of the stars from the sun's apex (or antapex). Otherwise, the circumstances are not essentially different from those just described. The density distribution of vector

terminals over the tangential plane remains unchanged. The origin for T_0, however, is not at $T=0$, but, for stars at the distance λ from the antapex, at some point on the circle of radius

$$T=V_0 \sin \lambda\ , \tag{13}$$

where V_0 is the solar motion. Vectors formed by connecting the new origin, defined by $T_0=0$, with the original vector terminals of the stars in question represent the uncorrected velocities. The distribution of density with respect to the origin $T_0=0$ is unsymmetrical, and different for different values of λ; but an equation of the form of (9) always holds, as in the case of corrected tangential velocity, provided we use for surface density a function $D_m(T_0)$ giving the *mean* density over the ring element of radius T_0 and width dT_0. From the conditions of the problem, $D_m(T_0)$ cannot be zero at the origin. Hence, as before, the distribution of the small velocities cannot be represented by a Gaussian function of T_0, and we must expect appreciable deviations from the Gaussian law to appear at about the same limit as before, namely, at $T_0=10$ km/sec.

For small values of T_0 the mean density over the ring element $2\pi T_0\,dT_0$ will be very nearly equal to the density at $T_0=0$, and, for any value of λ, can always be expressed by a series of the form (12). If anything, the convergence will be even more rapid than in the case of corrected tangential velocity, for the mean density in a series of small rings concentric about $T_0=0$ will change very slowly indeed; the asymmetrical distribution of density about $T_0=0$ is an actual advantage. But, even at the origin $T=0$, there can be no practical difficulty arising from the convergence. Hence, for a given λ, the distribution function for uncorrected tangential velocity also has the form of (5).

The integrated function for the whole sky follows at once by introducing the weighted sum, for all values of λ, of the various expressions for the surface density of the vector terminals. The coefficients change, but not the form of the function. Finally, therefore,

$$N(T_0)dT_0=(D_0+D_1T_0+D_2T_0^2+\ .\ .\ .\ .)T_0dT_0 \qquad (T_0 \lessgtr T_1), \tag{14}$$

where the limit T_1, in analogy with (11), is defined by

$$\log T_1 = \log T_0 - \frac{1}{h^2 \text{ Mod.}} . \qquad (15)$$

The quantity $h = 2.18$ is the modulus of the Gaussian function[1] which represents the distribution of $\log T_0$ for $T_0 > T_1$. The coefficients D_0, D_1, etc., are still unknown, and no data are available at present for accurately determining D_1, D_2, etc. The latter coefficients, however, must be very small as compared with D_0; hence, equation (14) is in substantial agreement with the empirical formula (2). More detailed examination of the numerical values used in *Contribution* No. 272 confirms this conclusion, and shows that the simple function[2]

$$N(T)dT = D_0 T dT \qquad (T \lessgtr T_1) \qquad (16)$$

is an excellent approximation.

From the manner in which (14) has been derived, it appears that $N(T)dT$ represents, not a probability, but the number of stars in a given (unit) volume having velocities between T and $T+dT$. To obtain the probability itself, let

n = total number of stars per cubic parsec of unit density
n_1 = number of stars per cubic parsec of unit density having $T \lessgtr T_1$
$n = \sigma n_1$

From (16)

$$n_1 = D_0 \int_0^{T_1} T dT = \tfrac{1}{2} D_0 T_1^2 .$$

Since the probability of a velocity between T and $T+dT$ is $N(T)dT/n$, the required expression is

$$P(T)dT = \frac{2TdT}{\sigma T_1^2} \qquad (T \lessgtr T_1) . \qquad (17)$$

The constant σ, or more conveniently $2/\sigma$, can be evaluated from the condition that at the limit T_1 the probability of a velocity

[1] See equation (72) *Mt. Wilson Contr.*, No. 272; *Astrophysical Journal*, **59**, 274, 1924.

[2] From here on the subscript zero, indicating *uncorrected* tangential velocity is omitted as a matter of convenience.

between T and $T+dT$ found from (17) must be the same as that given by the Gaussian function which holds for $T>T_1$. A second condition must also be satisfied, however, for the total probability over the interval $T=0$ to $+\infty$ must be unity. This necessitates the introduction of a second constant λ, and the conditions accordingly read

$$\frac{2}{\sigma}\frac{T_1 dT_1}{T_1^2}=\frac{\lambda h}{\sqrt{\pi}}e^{-h^2(\log T_1-\log T)^2}d(\log T_1)$$

$$\frac{2}{\sigma T_1^2}\int_0^{T_1}TdT+\frac{\lambda h}{\sqrt{\pi}}\int_{\log T_1}^{\infty}e^{-h^2(\log T_1-\log T)^2}d(\log T)=1$$

With the aid of (15), these reduce to

$$\frac{2}{\sigma}=\frac{\text{Mod. }\lambda h}{\sqrt{\pi}}e^{-t_1^2} \tag{19}$$

$$\frac{1}{\sigma}+\frac{\lambda}{\sqrt{\pi}}\int_{t_1}^{\infty}e^{-t^2}dt=1 \qquad t_1=-\frac{1}{h\text{ Mod.}}\ . \tag{20}$$

The adopted value of h from *Contribution* No. 272, (73) is 2.18; whence $2/\sigma=0.172$, and $\lambda=0.9805$. Strictly speaking, the numerical value of neither of these constants is required, although later we shall need the ratio

$$K=\frac{\sigma\lambda h\text{ Mod.}}{2\sqrt{\pi}}\ . \tag{21}$$

From (19) we have directly

$$\log K=\frac{1}{h^2\text{ Mod.}}=0.484\ . \tag{22}$$

4. By using (17) to the limit $T=T_1$ and the Gaussian function of $\log T$ for $T>T_1$, and proceeding as in *Contribution* No. 273, it is possible to derive a theoretical value of the mean parallax of the stars of magnitude m for any value of μ. When T is expressed in astronomical units per year, $T=\mu\rho$ and $T_1=\mu\rho_1$; and, since $\rho=e^{-a'x}$, where $a'=1/\text{Mod.}$ and $x=-\log\rho$, (17) becomes

$$P(T)dT=\frac{2e^{-2a'x}d\mu}{\sigma\rho_1^2\mu} \qquad (\rho \gtrless \rho_1)\ . \tag{23}$$

The Gaussian expression which holds for larger values of T can be written in the form used in *Contribution* No. 273 [equations (8) and (23) to (27)],

$$P(\tau)d\tau=\frac{\lambda h}{\sqrt{\pi}}e^{-h^2(\tau-\bar{\tau})^2}d\tau=\frac{\text{Mod. } h\lambda}{\sqrt{\pi}\,\mu}e^{\alpha_2+\beta_2x+\gamma_2x^2}d\mu \qquad (\rho>\rho_1)\,, \tag{23a}$$

where

$$\tau=\log T\,, \qquad \bar{\tau}=\log \underline{T}\,.$$

$P(T)dT$, like $P(\tau)d\tau$, is an exponential function of x; and in analogy with equation (17) of that paper, we have

$$N_1(x)dx=K_1'e^{\alpha'+\beta'x+\gamma'x^2}dx \qquad (x\gtrless x_1)\,. \tag{24}$$

This equation expresses the number of stars having magnitudes between m and $m+dm$, and proper motions between μ and $\mu+d\mu$, for which the characteristic $x=\log x$ lies between x and $x+dx$. It applies only to stars having $T\lessgtr T_1$, as indicated by the condition $x\gtrless x_1$, where $x_1=-\log\rho_1$, and $\rho_1=T_1/\mu$. The number of stars having $T>T_1$, but otherwise with the same characteristics as above, is given by equation (17) of *Contribution* No. 273:

$$N(x)dx=K_1e^{\alpha+\beta x+\gamma x^2}dx \qquad (x<x_1)\,. \tag{25}$$

The logarithm of the geometrical mean parallax has therefore the form

$$\log \underline{\pi}=\bar{x}=\frac{\int_{+\infty}^{x_1}xN_1(x)dx+\int_{x_1}^{-\infty}xN(x)dx}{\int_{+\infty}^{x_1}N_1(x)dx+\int_{x_1}^{-\infty}N(x)dx} \tag{26}$$

while the right-hand member of this expression with π written in place of the factor x in the integrals of the numerator gives directly the arithmetical mean $\bar{\pi}$.

5. Equation (26) does not lend itself readily to an algebraic expression of $\log \underline{\pi}$ as a function of m and μ except for the limiting value $\mu=0$. For this case it reduces to a very simple form.

Since $\mu\rho_1=T_1$, $x_1=-\log\rho_1$ approaches $-\infty$ as μ approaches 0. Hence for $\mu=0$ the second integral in both numerator and denominator of (26) is zero. Further, the quotient of the remaining integrals

then becomes simply the mean of the values of x distributed according to the Gaussian curve (24), so that

$$\log \pi = -\frac{\beta'}{2\gamma'} \qquad (\mu = 0) \tag{27}$$

Again, by analogy with the development in *Contribution* No. 273,

$$\beta' = \beta_1 + \beta_2' = \beta_1 - 2a'$$
$$\gamma' = \gamma_1 + \gamma_2' = \gamma_1$$

where β_1 and γ_1 are defined by (21) and (22) of that paper, and $\beta_2' = -2a'$ and $\gamma_2' = 0$ are the coefficients of the first and second powers of x in (23) above. Hence

$$\beta' = \beta_1 - 2a' = -5a' - k + 5q + 10rm \tag{28}$$
$$\gamma' = \gamma_1 = l + 25r \qquad a' = 1/\text{Mod.} \tag{29}$$

Equations (27) to (29) give the geometrical mean parallax for $\mu = 0$ as a function of m and the coefficients q and r of the luminosity function, and k and l of the density function. The arithmetical mean can be found by replacing the coefficient of a' in (28) by 4.5.

If the formula for the latter case be compared with equations (4), (5), and (6) of *Contribution* No. 281, it will be seen that

$$\log \bar{\pi}_m - \log \bar{\pi}_{m,0} = -\frac{a'}{\gamma_1} = -\frac{1}{\text{Mod.}\ (l + 25r)}\,. \tag{30}$$

Hence, for stars of magnitude m, the ratio of the mean parallax for all proper motions collectively to the mean parallax for $\mu = 0$ depends only on l and r, the dispersion coefficients of the density and luminosity functions. Further, the difference expressed by (30) is, by (8) of *Contribution* No. 281, equal to $1/\kappa^2$Mod., where κ is the modulus of the Gaussian distribution function of $\log \pi_m$ (stars of magnitude m, all proper motions collectively).

These results of course depend upon the applicability of the quadratic exponentials used for the luminosity and density functions. The limitations, which have been discussed at length in *Contributions* Nos. 273 and 281, are, briefly, that the luminosity function of Kapteyn and van Rhijn can be used for $M < 7.5$ (international scale), and that for a given pair of values of k and l

the density function holds for a range of about 1.6 in log ρ. The restrictions upon the use of (27) to (29) or of (30) are the same as those pointed out in *Contribution* No. 281, that is, to stars brighter than eleventh or twelfth apparent magnitude, and to values of k and l corresponding closely to the median log ρ of the group in question.

Eventually these formulae should afford a useful control upon the accuracy of the measured parallaxes for very small proper motions. At present, however, the values of k and l are not known with sufficient precision to give information of much importance. As an illustration, however, equation (30) has been used with the data in Tables I and IV of *Contribution* No. 281 to calculate the values of $\bar{\pi}_{m,0}$ to $m=6$. The results, with van Rhijn's values for $\mu=0$ from Table 19, *Groningen Publication* No. 34, for comparison, are as follows:

m	1	2	3	4	5	6
Equation (30)	0".0105	0".0076	0".0058	0".0045	0".0035	0".0027
van Rhijn....	(0.0097)	0.0080	0.0066	0.0054	0.0045	0.0037

The two series of mean parallaxes are of the same order of magnitude; beyond $m=2$ the differences emphasize the fact that we are dealing with a region in which the stellar density is still imperfectly known.

6. Some doubt may remain as to the adequacy of (16) as an expression for the distribution function for tangential velocity. As already stated, the evidence for its sufficiency seems conclusive; but it is worth remarking that for the limiting case $\mu=0$, higher powers of T included in (14) would contribute nothing to the result, irrespective of the values of the coefficients D_1, D_2, etc. The neglected terms would give rise to additional integrals in (26), similar to the first in both numerator and denominator; but these integrals would appear multiplied by powers of μ, and hence disappear for $\mu=0$.

The result for this case is also independent of the form of the distribution function for $T>T_1$. Practically, this is only another

way of saying that stars of zero tangential velocity alone can have zero proper motions. If T is finite, μ can be zero only for $\rho=\infty$; but the density function is zero for $\rho=\infty$. Hence stars having finite values of T contribute nothing to the mean parallax for $\mu=0$.

Equations (22) to (30) are therefore rigorous, except in so far as the quadratic exponentials used to represent the distribution of luminosity and density may prove defective. There is little difficulty on this score, however. We are dealing with a limited range, both in luminosity and distance. Except in the case of large values of m, the absolute magnitudes all fall on the ascending branch of the curve of Kapteyn and van Rhijn; and, as shown in *Contribution* No. 281, the quadratic exponential for density is adequate, provided the coefficients are properly chosen. These, there is reason to believe, will soon be known with an accuracy sufficient to calculate the limiting parallaxes with precision to the sixth or seventh magnitude.

7. The point at which the general equation (26) begins to show appreciable deviations from the simple formula (1) can be found only by deriving numerical results for different values of m and μ. Extensive comparisons are scarcely justified by our present knowledge of the constants involved. Nevertheless, some indication is desirable, and a few results are given below.

To adapt the formulae to calculation, (24) and (25) may be written

$$N_1(x)dx=Ce^{\beta_1 x+\gamma_1 x^2}dx\left(\frac{2}{\sigma\rho_1^2}\right)e^{-2a'x} \qquad (x \geqq x_1) \tag{31}$$

$$N(x)dx=Ce^{\beta_1 x+\gamma_1 x^2}dx\left(\frac{\text{Mod. }\lambda h}{\sqrt{\pi}}\right)e^{\alpha_2+\beta_2 x+\gamma_2 x^2} \qquad (x<x_1) \tag{32}$$

where

$$C=\frac{-4\pi a' e^{\alpha_1}dm\,d\mu}{\mu}$$

is a common factor which disappears on substituting into (26). The factors following dx in (31) and (32), and the quotient $d\mu/\mu$ included in C, represent the distribution functions for velocity,

(23) and (23*a*). With the aid of (31) and (32), together with (21), (26) becomes

$$\bar{x}=\frac{\int_{+\infty}^{x_1} xe^{(\beta_1-2a')x+\gamma_1x^2}dx+K\rho_1^2\int_{x_1}^{-\infty} xe^{\alpha_2+\beta x+\gamma x^2}dx}{\int_{+\infty}^{x_1} e^{(\beta_1-2a')x+\gamma_1x^2}dx+K\rho_1^2\int_{x_1}^{-\infty} e^{\alpha_2+\beta x+\gamma x^2}dx} \tag{33}$$

where

$$\beta=\beta_1+\beta_2\,,\qquad \gamma=\gamma_1+\gamma_2\,. \tag{34}$$

Introducing

$$x_0=-\frac{\beta_1-2a'}{2\gamma_1}\qquad x_\mu=-\frac{\beta}{2\gamma}\,, \tag{35}$$

(33) becomes

$$\bar{x}=\frac{e^{-\gamma_1x_0^2}\int_{+\infty}^{x_1} xe^{\gamma_1(x-x_0)^2}dx+K\rho_1^2e^{\alpha_2-\gamma x_\mu^2}\int_{x_1}^{-\infty} xe^{\gamma(x-x_\mu)^2}dx}{e^{-\gamma_1x_0^2}\int_{-\infty}^{x_1} e^{\gamma_1(x-x_0)^2}dx+K\rho_1^2e^{\alpha_2-\gamma x_\mu^2}\int_{x_1}^{-\infty} e^{\gamma(x-x_\mu)^2}dx} \tag{36}$$

which is easily expressed as a function of the probability integral

$$P_a^b=\frac{2}{\sqrt{\pi}}\int_a^b e^{-t^2}dt\,. \tag{37}$$

The final form is

$$\log \underline{\pi}=\bar{x}=\frac{-\dfrac{e^{-t_1^2}}{\sqrt{\pi}\,h_1}+x_0P_\infty^{t_1}+C_1\left(\dfrac{e^{-t_2^2}}{\sqrt{\pi}\,h_2}+x_\mu P_{t_2}^{-\infty}\right)}{P_\infty^{t_1}+C_1P_{t_2}^{-\infty}} \tag{38}$$

in which

$$\left.\begin{array}{lll} t_1=h_1(x_1-x_0) & t_2=h_2(x_1-x_\mu) & \\ h_1=\sqrt{-\gamma_1} & h_2=\sqrt{-\gamma} & C_1=K_0h_1/h_2 \end{array}\right\}\,. \tag{39}$$

$$K_0=K\rho_1^2e^{\alpha_2+\gamma_1x_0^2-\gamma x_\mu^2}\,. \tag{40}$$

The limiting tangential velocity T_1, which marks the transition from the frequency function (23) to (23*a*), is given by (15). Since $T_1=\mu\rho_1$ and $x=-\log\rho_1$, and, further, since $1/h^2\text{Mod.}=0.484$,

$$x_1=\log\mu-\log\underline{T}+0.484\,. \tag{41}$$

The geometrical mean tangential velocity $\underline{T}$ is a function of the absolute magnitude M. For the present purpose it is sufficient to use the linear relation

$$\log T = 0.84 + 0.05\, M \tag{42}$$

where $M = m + 5x$. The stars with which we are concerned are all intrinsically bright, and the coefficients of (42) have been chosen so that the data of Table VI, *Contribution* No. 272, are well represented between $M = -3$ and $+3$ (international scale).[1] Equation (41) thus becomes

$$x_1 = \log \mu - 0.36 - 0.05m - 0.25x\ . \tag{43}$$

Further, with the aid of (22), (40) can be written

$$\log K_0 = 0.484 - 2x_1 + \text{Mod.}\ (\alpha_2 + \gamma_1 x_0^2 - \gamma_1 x^2)\ . \tag{44}$$

The constant K_0 depends upon x_1, which in turn depends on x. Strictly speaking, the term in x should have been retained under the integral sign and included in the integration; practically, this complicates the calculation without affording any important gain in precision. For a general survey of the numerical relations it is quite sufficient to replace x in (43) by $\bar{x}$, for which an approximation is easily found.

There remains now only the evaluation of α_2, β_2, and γ_2. For the immediate purpose this is best done by equating the coefficients of the exponents in (23*a*), which are connected by the identity

$$-h^2(\tau - \bar{\tau})^2 = \alpha_2 + \beta_2 x + \gamma_2 x^2\ . \tag{45}$$

Since

$$\tau - \bar{\tau} = \log \mu - s - tm - x - 5tx$$

where s and t are the coefficients in the right-hand member of (42), and since $h^2 = 4.75$, we have finally,

$$\left.\begin{aligned} \alpha_2 &= -4.75\ (\log \mu - 0.05m - 0.84)^2 \\ \beta_2 &= +4.75 \times 2.5\ (\log \mu - 0.05m - 0.84) \\ \gamma_2 &= -7.42 \end{aligned}\right\}\ . \tag{46}$$

[1] The maximum residual is 0.02. In verifying the comparison it must be remembered that T in (42) is expressed in astronomical units per year, and that M is referred to Kapteyn's zero point.

This supplies all the details necessary for the calculation of $\bar{x}$ by (38). The auxiliary formulae required are (46), (28), and (29), (34), (35), (43), (44), and (39).

To obtain an expression for $\bar{\pi}$ it is only necessary to replace the factors x by π in the numerator of (36), and use the relation $\pi = e^{a'x}$. The expression is then easily reduced to a function of the probability integral (37), giving

$$\bar{\pi} = \frac{\bar{\pi}_0 P^{t'_i}_{\infty} + \bar{\pi}_\mu C_1 P^{-\infty}_{t'_i}}{P^{t'_i}_{\infty} + C_1 P^{-\infty}_{t'_i}} , \tag{47}$$

where $\bar{\pi}_0$ and $\bar{\pi}_\mu$ are defined by

$$\log \bar{\pi}_0 = x_0 - \frac{a'}{4\gamma_1} \qquad \log \bar{\pi}_\mu = x_\mu - \frac{a'}{4\gamma} \tag{48}$$

$$t'_1 = t_1 - \frac{a'}{2h_1} \qquad t'_2 = t_2 - \frac{a'}{2h_2} \, . \tag{49}$$

By comparing (27) and (35), it will be seen that x_0 is nothing but the logarithm of the geometrical mean parallax for the limiting case $\mu = 0$. The quantity x_μ, on the other hand, is the value for the proper motion μ (and magnitude m) corresponding to the assumption that the Gaussian frequency function (23*a*) holds for all values of T. See equation (18), *Contribution* No. 273. Similarly, $\bar{\pi}_0$ and $\bar{\pi}_\mu$ defined by (48) are the arithmetical mean parallaxes for these two special cases. From (39), (49), and (43) it appears that all the t''s equal $-\infty$ for $\mu = 0$. For this case $\log \underline{\pi}$ from (38) and $\bar{\pi}$ from (47) reduce to x_0 and $\bar{\pi}_0$, respectively, as they should. On the other hand, the t''s all become positive as soon as μ has attained a value of a few thousandths of a second of arc, so that the results given by (38) and (47) become sensibly equal to x_μ and $\bar{\pi}_\mu$, respectively, for rather small values of μ.

8. To illustrate these results, equations (38) and (47) have been used to calculate the mean parallaxes of stars of the fifth apparent magnitude for a few values of μ. The resulting logarithms for the geometrical and arithmetical means are in the second and third

columns of Table I. The fifth column gives the values of log $\underline{\pi}$ calculated from the second of (35), designated by x_μ above. These results correspond to the assumption of a Gaussian distribution of log T for all values of T, and are those given by Kapteyn's formula, equation (1) (modified to give the geometrical mean). The corrections necessary to reduce to the revised formula (38) are shown in the sixth column.

TABLE I

MEAN PARALLAXES FOR SMALL PROPER MOTIONS ($m=5$)

μ	log $\underline{\pi}$ Eq. (38)	log $\overline{\pi}$ Eq. (47)	log $\overline{\pi}$ −log $\underline{\pi}$	log $\underline{\pi}$ Eq. (35)	Corr.	$\overline{\pi}$ Eq. (47)	$\overline{\pi}$ v. R.
0".000....	−2.693	−2.528	+0.165	−∞	+∞	0".0030	0".0045
.005....	2.415	2.327	0.088	−2.495	+0.080	.0047	.0053
.01.....	2.285	2.203	0.082	2.330	0.045	.0063	.0061
.02.....	2.144	2.075	0.069	2.168	0.024	.0084	.0077
0.04.....	−1.996	−1.932	+0.064	−2.005	+0.009	0.0117	0.0106

From the column of corrections it is seen that the failure of the Gaussian law for the distribution of log T begins to be appreciable (1 per cent) in the case of the stars of the fifth magnitude at $\mu=0''.03$ or $0''.04$. Since the coefficient of m in (43) and (46) is small, the limit for stars of other magnitudes will not be very different.

The differences in the fourth column of the table bring to light a point of some interest, namely, that the dispersion in log π for a given m and μ increases appreciably with decreasing proper motion. Finally, the last two columns show the agreement of the results from equation (47) with van Rhijn's tables in *Groningen Publications* No. 34. This, at best, however, can be only approximate, because the calculation has been made with constant values of the density coefficients k and l. This affects the mean parallaxes considerably, and, incidentally, accounts for the fact that for $\mu=0$, $\overline{\pi}$ in Table I is $0''.0030$, whereas equation (30), with more appropriate values of k and l, gave $0''.0035$. The value of μ for which the corrections to Kapteyn's formula become appreciable is not much affected, however.

For purposes of interpolation, simple formulae representing the values of log $\overline{\pi}$ and log $\underline{\pi}$ for small values of μ are desirable. What

will suffice depends upon the precision required. Expressions for log $(\underline{\pi}+k_1)$ and log $(\bar{\pi}+k_2)$ as quadratic functions of log μ, with terms to include the dependence on m, seem to be well adapted to the purpose. The form is substantially that of the empirical relation of van Rhijn, and has the advantage that the quantities k_1 and k_2 are equal, respectively, to $-\underline{\pi}$ and $-\bar{\pi}$ for $\mu=0$, which will sometimes facilitate the determination of the constants.

Mount Wilson Observatory
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THE SYSTEM OF 61 μ ORIONIS

By EDWIN B. FROST AND OTTO STRUVE

ABSTRACT

Elements of the orbit of the spectroscopic binary.—Observations of the star 61 μ Orionis were begun at the Yerkes Observatory in 1905, and a table of the measured velocities from 124 plates obtained in the intervening years is given. The period is found to be $4^{d}447$, and $a \sin i = 2{,}000{,}000$ km/sec. The eccentricity is found to be 0.01, indicating a circular orbit. The velocity in the line of sight of the center of mass varied from 44 km/sec. in 1907 to 37 km/sec. in 1915, and back to 43 km/sec. in 1922, indicating the presence of a third body.

Tentative elements of the orbit of the visual binary.—In 1914, Aitken found that μ Orionis was a visual double, the fainter component being of magnitude 6.6, separation $0''.36$. A table of seven observations of this companion is given from the measures of Aitken and Van Biesbroeck. Tentative elements derived for the orbit of the visual binary are: $P = 18$ years; $\gamma = 40.8$ km/sec.; $K = 4.0$ km/sec.; $\omega = 98°$; $e = 0.6$; $T = 1911.7$; $a \sin i = 300{,}000{,}000$ km. The indications are that $\sin i$ is in the neighborhood of unity, so that the distance of the visible companion from the spectroscopic binary is approximately two astronomical units. Assuming the parallax to be $0''.03$ and that $\sin i$ is nearly equal to unity, the mass of the visible companion is found to be 1.2 $\odot$, while the masses of the components of the spectroscopic binary are 2.1 $\odot$ and 1.7 $\odot$ respectively. Further observation through an entire period of the visual binary is considered necessary.

This star, which is both a visual and a spectroscopic binary, has the following characteristics: its position for 1925 is $\alpha = 5^{h}58^{m}.2$, $\delta = +9°39'$; its proper motion is given in Boss's *Preliminary Catalogue* as $+0^{s}.0012$ and $-0''.029$; the *Henry Draper Catalogue* gives its visual magnitude as 4.19 and its photographic magnitude as 4.25, whence its color index is +0.06; its spectral type is A2. The letter *s* should be added to the designation of type, as the lines are particularly sharp and narrow and suggestive of class F. Its spectrum, like that of α Cygni, ϵ Aurigae, and a very few other bright stars, should, perhaps, be considered as on a collateral branch between A2 and F. It was placed upon the parallax program of the 40-inch telescope in 1907, and the measurements by S. A. Mitchell from eight plates gave a parallax of $+0''.036 \pm 0''.016$. Burnham lists a companion to this star of the 14th mag. at a distance of $17''$ (βG.C. 3111), but it now appears to be only an optical system, and nothing further in this paper will have any reference to that object.

The star was found to be a spectroscopic binary by Frost in 1906[1] from an examination of two plates, taken with a dispersion of

[1] *Astrophysical Journal*, **23**, 266, 1906.

three prisms, which showed a range of 35 km/sec. The lines are particularly well suited for accurate measurement, and this fact, together with the large range, has caused the star to be used as an excellent example of a spectroscopic binary either in an engraving or on a lantern slide. Suspicions in the measurements of the plates were entertained that traces of a second component might be visible, but a re-examination of these cases led to the conclusion that they were unreal and could have no relation to the objects under discussion in this paper. The star has been kept under observation at intervals since its detection, and graphical determinations of the orbit were made by Frost in 1912, which showed an eccentricity of 0.04 and a rather large positive value for the velocity of the center of mass of +45 km/sec. The period was at first thought to be $0^{d}.77$, or about one-sixth of its true value of $4^{d}.447$ found somewhat later, and this incorrect value was communicated to Campbell for his *Second Catalogue of Spectroscopic Binaries.* The other elements were found to be as follows: $K = 30.0$ km/sec.; $\omega = 180°$; $a \sin i = 2{,}000{,}000$ km. A least-squares solution brought the eccentricity down to 0.01.

The spectrograms of this star were examined for systematic variation of velocity with wave-length in 1909 to test the existence of a dispersion in space, which was asserted by some astronomers at that time; no evidence was found to support such a hypothesis.[1]

Another graphical orbit, determined from observations a few years later, gave evidence indicative of a change in the velocity of the system which would imply the presence of a third body. In view of this fact, we were particularly interested in the announcement made by Aitken in 1914[2] of the discovery of a close visual companion, of magnitude 6.6, to μ Orionis, with a position angle of $32°.0$ and distance of $0''.36$. Professor Aitken has very kindly communicated to us his observations of this star, and Professor Van Biesbroeck also made occasional observations of it, which are contained in Table I.

From this, it will appear that the position angle has changed only slightly while the distance has diminished appreciably, in fact

[1] *Publications of the Astronomical and Astrophysical Society of America*, **1**, 324, 1909.

[2] *Lick Observatory Bulletin*, **8**, 93, 1914.

to nearly one-half of its value in 1917. It will be shown a little later that the changes in the γ value indicate a period of about twenty years.

TABLE I

Date	Position Angle	Dist.	Nights	Observer	Remarks
1914.74	32°0	0″36	3	Aitken	....
1917.41	20.4	.38	2	Aitken	Difficult
1918.11	16.4	.31	1	Aitken	Very difficult
1919.97	25.8	.38	1	Van Biesbroeck	....
1920.51	15.8	.32	2	Aitken	....
1921.80	22.7	.30	2	Aitken	Very difficult
1924.24	18.0	0.20	3	Van Biesbroeck	....

In Table II the observations are arranged in five groups, each covering two or three seasons of observations. The measurements of the second group of plates were of much lower weight than those of the others, as we were at that time having trouble with astigmatism of the camera lens due to the cement. Further, the plates at that time were taken at short intervals to ascertain the period with certainty. The probable errors of a single determination of velocity for each of the five groups are ± 2.5 km/sec.; ± 5.3 km/sec.; ± 2.2 km/sec.; ± 1.9 km/sec.; and ± 2.6 km/sec. On account of the excellence of this spectrum, it was used for practice measurements by several who were acquiring experience in measurement of spectra, and in some instances these measures have been combined in Table II. For Group IV a complete least-squares solution for elliptic elements has recently been made by Struve, with the result that the eccentricity was 0.01. Accepting this as indicating a circular orbit, he then made a solution for a circular orbit for all five groups. The results are given in Table III. T denotes here the time of minimum velocity.

The curve in Figure 1 represents the observations in Group IV. The corrections to the values of K thus found for the different groups do not exceed their probable errors, so that we may safely conclude that the velocity in the orbit is constant. It also appears from Table III that the change in γ is confirmed, and the smallness of the probable errors gives us great confidence in these values.

TABLE II

Plate	G. M. T.	Observed By	Measured By	Quality	Vel. km/sec.	O.−C. km/sec.
II B 25.....	1905 Nov. 24.787	F, B, S	F	g.	65.3	− 6.8
I B 624.....	Dec. 9.754	F, S	F	v.g.	45.6	− 4.6
I B 650.....	Dec. 25.689	F	F	g.	57.6	+ 3.1
B 623.....	1906 Jan. 5.705	F, S	F, L	g.	38.2	− 0.8
B 631.....	Jan. 8.790	F, B, S	F	f.	71.5	2.0
I B 669.....	Jan. 29.592	F, S	F	p.	21.2	5.0
B 641.....	Feb. 9.625	F, S	F	g.	63.6	1.1
B 646.....	Mar. 16.500	B, S	F	f.	73.2	− 0.8
I B 898.....	Oct. 31.950	B, S	F	v.g.	22.5	+ 6.2
I B 906.....	Nov. 1.888	I, S	F	g.	49.9	1.5
B 683.....	Dec. 23.801	Fox, S	U	g.	18.4	2.6
B 691.....	Dec. 25.736	Fox	L, My	p.	66.9	0.8
I B 938.....	1907 Jan. 4.729	F, S	F	p.	69.3	4.3
I B 946.....	Jan. 5.808	Fox, S	F	g.	25.2	+ 1.7
I B 955.....	Jan. 25.622	F, S	F	f.	53.2	− 4.5
I B 960.....	Jan. 25.872	B, S	F	g.	68.9	+ 2.7
I B 967.....	Feb. 2.600	Fox, S	F	g.	21.7	− 0.4
I B 971.....	Feb. 2.778	Fox, S	F	p.	24.3	± 0.0
I B 973.....	Feb. 8.586	F, S	L, F	f.	80.3	+ 6.9
I B 977.....	Feb. 8.811	B, S	F	p.	71.0	− 2.9
I B 978.....	Feb. 9.588	F, S	L, F	g.	54.8	0.7
B 706.....	Mar. 8.562	F, S	L, F	f.	41.7	1.5
B 713.....	Mar. 23.613	F, S	F	p.	22.0	− 2.9
II B 94.....	Mar. 29.588	B, S	L	f.	74.1	+ 0.2
II B 101.....	Mar. 31.572	F	F	f.	19.7	2.8
I B 1172.....	Sept. 21.915	B, S	F	g.	35.0	+10.5
II B 188.....	Nov. 8.831	L, B, S	L, My	p.	18.9	− 1.5
II B 201.....	Nov. 11.969	L, S	L, My	g.	67.5	− 0.8
B 732.....	Nov. 15.846	F, S	*	g.	77.9	+ 3.0
I B 1265.....	Dec. 4.673	F, S	L, F	p.	42.0	− 9.9
I B 1266.....	Dec. 4.699	F, S	L, F	p.	48.0	− 2.8
I B 1267.....	Dec. 4.774	F, S, B	L, F	p.	51.0	+ 3.4
I B 1268.....	Dec. 4.808	B, S	L, F	f.	45.5	− 0.7
I B 1269.....	Dec. 4.874	B, S	L	p.	31.0	12.4
I B 1270.....	Dec. 4.905	B, S	L	p.	26.0	16.1
I B 1271.....	Dec. 4.938	B, S	L	g.	34.0	− 6.7
I B 1275.....	Dec. 6.603	F, S	L, U, F	f.	35.4	+ 8.3
I B 1279.....	Dec. 6.805	L, S	L, F	v.g.	35.8	1.2
I B 1282.....	Dec. 6.966	L, S	L, F	f.	52.9	11.7
I B 1287.....	Dec. 11.659	L, S	L, F	p.	59.0	7.4
II B 206.....	1908 Mar. 9.633	L, S	F	g.	56.2	3.6
II B 213.....	Mar. 15.578	F, L, S	F	p.	68.2	+ 2.1
I B 3901.....	1914 Nov. 16.749	B, S	F	g.	52.1	− 7.2
I B 3903.....	Nov. 16.872	My, S	F	f.	58.0	− 2.0
I B 3910.....	Nov. 20.771	My, S	F	g.	67.1	+ 0.2
I B 3912.....	Nov. 20.880	B, S	F	v.g.	66.2	0.2
I B 3918.....	Nov. 24.718	F, S	F	g.	65.8	+ 3.2
I B 3924.....	Nov. 27.706	My, S	F	g.	4.9	− 6.2
I B 3926.....	Nov. 27.923	B, S	F	f.	14.3	+ 0.5
I B 3928.....	Dec. 4.763	My, S	F	g.	52.2	0.4
I B 3933.....	Dec. 14.653	B, S	F	f.	21.8	8.0
I B 3936.....	Dec. 14.870	F, S	F	p.	11.4	+ 0.2

* Buch, St, Leo, U, L, G, F, σ.

TABLE II—*Continued*

Plate	G. M. T.	Observed By	Measured By	Quality	Vel. km/sec.	O.−C. km/sec.
I B 3951.....	1914 Dec. 25.660	B, S	F	f.	53.6	− 3.8
I B 3955.....	Dec. 25.849	F, S	F	p.	66.2	+ 3.6
I B 3960.....	1915 Jan. 4.683	F, S	F	g.	57.7	− 1.2
I B 3964.....	Jan. 8.644	My, S	U	f.	66.9	− 0.2
I B 3967.....	Jan. 8.837	B, S	F	g.	67.7	+ 2.3
I B 3969.....	Jan. 12.510	F, B, My	U	f.	55.5	− 3.7
I B 3970.....	Jan. 12.544	B	U	v.g.	58.2	2.0
I B 3971.....	Jan. 12.575	B, S	U	g.	60.1	0.9
I B 3972.....	Jan. 12.609	S, F	U	g.	61.4	− 0.4
I B 3973.....	Jan. 12.643	S, F	U	g.	64.1	+ 1.5
I B 3974.....	Jan. 12.674	S, F	U	g.	60.1	− 3.2
I B 3975.....	Jan. 12.704	S, F	U	f.	65.7	+ 1.8
I B 3976.....	Jan. 12.749	S, F	U	p.	66.3	+ 1.6
I B 3977.....	Jan. 12.794	S, F, B	U	p.	61.0	− 4.4
I B 3986.....	Jan. 26.672	S	F	f.	66.7	+ 2.0
I B 3987.....	Jan. 26.704	S	F	g.	68.7	4.6
I B 3988.....	Jan. 26.750	S	F	g.	63.3	0.1
I B 3989.....	Jan. 26.801	S, B	F	g.	64.2	+ 2.1
I B 4024.....	Mar. 1.703	B	F	g.	28.0	− 3.1
I B 4032.....	Mar. 8.683	F, S	F	g.	36.0	+ 5.6
I B 4038.....	Mar. 9.706	My, S	F	g.	6.0	− 1.4
I B 4073.....	Mar. 29.641	B, S	F	f.	61.0	− 1.2
I B 4080.....	Mar. 30.637	F, S	F	g.	46.0	+ 3.7
I B 4418.....	1916 Feb. 29.604	S, H	Lz, Mk	g.	34.6	− 5.3
I B 4423.....	Mar. 3.694	B, S	Lz, Mk	v.g.	9.4	+ 1.2
I B 4435.....	Mar. 10.654	B, S	Lz, Mk	v.g.	69.4	+ 2.5
I B 4705.....	1916 Dec. 1.691	Mk, S	Mk	g.	52.9	− 2.0
I B 4714.....	Dec. 13.657	B, S	Mk	g.	13.0	+ 3.9
I B 4731.....	1917 Jan. 1.590	B, S	Mk	g.	46.5	+ 0.7
I B 4733.....	Jan. 1.702	B, S	Mk	v.g.	47.1	− 3.2
I B 4735.....	Jan. 1.810	B, S	Mk	g.	49.2	− 5.2
I B 4742.....	Jan. 3.748	Mk, S	Mk	g.	34.3	+ 0.3
I B 4747.....	Jan. 5.668	B, S	Mk	g.	28.9	− 1.4
I B 4753.....	Jan. 10.620	B	Mk	g.	49.2	2.0
I B 4756.....	Jan. 15.601	B	Mk	g.	65.8	− 0.5
I B 4759.....	Jan. 15.803	B, Mk	Mk	f.	70.4	+ 2.1
I B 4765.....	Jan. 17.825	B, Mk	Mk	f.	13.3	+ 2.7
I B 4770.....	Jan. 19.707	B	Mk	g.	55.0	− 3.0
I B 4788.....	Jan. 29.592	B, S	Mk	g.	67.2	+ 2.9
I B 4795.....	Feb. 2.583	B, S	Mk	g.	66.6	− 1.8
I B 4800.....	Feb. 5.596	Mk, S	Mk	g.	22.9	+ 0.4
I B 4866.....	Apr. 9.586	B, S	U	p.	49.0	− 2.9
I B 5133.....	Dec. 21.850	B, S	Wk	g.	11.8	+ 1.4
I B 5160.....	1918 Jan. 14.641	Wk, S	Wk	f.	45.4	− 1.1
I B 5177.....	Jan. 28.626	Wk, S	Wk	v.g.	67.2	+ 1.0
I B 5179.....	Jan. 28.764	Wk, S	Wk	g.	70.3	2.4
I B 5191.....	Feb. 4.603	Wk, S	Wk	g.	10.8	2.2
I B 5206.....	Mar. 1.599	Wk, S	Bk	g.	63.9	4.4
I B 5229.....	Apr. 1.579	Wk, S	Bk	g.	66.3	+ 2.7
I B 5236.....	Apr. 8.547	B, Wk	Bk	f.	18.5	− 3.8
I B 5423.....	Nov. 25.775	B, S, Wk	Bk	g.	21.8	− 3.6
I B 5442.....	1919 Jan. 27.720	Wk, S	Bk	g.	51.7	− 1.1
I B 5460.....	Feb. 17.611	Wk, S	Bk	p.	14.8	+ 1.8

TABLE II—*Continued*

Plate	G. M. T.	Observed By	Measured By	Quality	Vel. km/sec.	O.−C. km/sec.
I B 5706.....	1920 Mar. 8.632	Bk, S	Bk	g.	68.3	− 1.7
I B 6043.....	Oct. 8.936	Bk, S	Bk	f.	43.6	+ 2.0
I B 6082.....	Dec. 6.838	B, S	Bk	v.g.	14.6	1.6
I B 6099.....	1921 Jan. 7.745	Bk, S	Bk	g.	30.7	+ 1.3
I B 6113.....	Jan. 28.610	Bk, S	Bk	g.	13.6	− 7.1
I B 6118.....	Jan. 31.583	Bk, S	Bk	f.	67.3	5.0
I B 6129.....	Feb. 14.614	Bk, S	Bk	f.	53.1	1.0
I B 6135.....	Feb. 18.516	Bk, S	Bk	g.	68.1	2.3
I B 6141.....	Mar. 25.612	Bk, S	Bk	f.	71.3	− 0.7
I B 6334.....	1922 Jan. 6.734	σ, S	σ	g.	13.8	+ 0.6
I B 6344.....	Jan. 9.718	σ, S	σ	v.g.	55.1	+ 0.7
I B 6353.....	Jan. 13.703	σ, S	σ	g.	64.8	− 4.1
I B 6369.....	Jan. 23.761	σ, S	σ	g.	25.4	0.6
I B 6464.....	Apr. 7.565	σ, S	σ	g.	67.9	− 3.0
B 998.....	Dec. 4.726	σ, S	σ	g.	62.7	+ 4.4
I B 6811.....	1923 Mar. 16.615	σ, S	σ	g.	71.6	1.8
I R 6852.....	Apr. 4.591	B, S	σ	g.	28.8	1.6
I R 7331.....	1924 Mar. 8.628	σ, S	σ	v.g.	19.6	3.5
I R 7336	Mar. 15.603	σ, S	σ	v.g.	72.3	+10.3

In Table II the names of the observers and measurers are indicated as follows: B=S. B. Barrett; Bk=Miss D. W. Block; Buch=D. Buchanan; Fox=P. Fox; F=E. B. Frost; G=Miss Vera M. Gushee; H=E. P. Hubble; I=N. Ichinohe; L=O. J. Lee; Lz=J. Lemkowitz; Leo=F. C. Leonard; My=C. A. Maney; Mk=G. S. Monk; St=H. T. Stetson; σ=O. Struve; S=F. R. Sullivan; U=R. S. Underwood; Wk=Miss E. W. Wickham.

In the column for quality of the plate, f.=fair, g.=good, p.=poor, v.=very.

TABLE III

Group	Epoch	P In Days	T In J. D.	K In km/sec.	γ In km/sec.
I..........	1906.9	4.44746	2,423,863 .194±0.016	30.4±0.7	+44.0±0.5
II.........	1908.0	4.44746	2,423,863 .358±0.036	28.8±2.6	44.3±1.3
III........	1915.2	4.44746	2,423,863 .144±0.015	30.1±0.6	37.3±0.5
IV........	1917.5	4.44746	2,423,863 .141±0.013	30.5±0.5	39.3±0.4
V.........	1921.7	4.44746	2,423,863 .174±0.021	30.8±0.8	+42.7±0.6

Accordingly, we are in a position to derive rough elements for the visual binary system which give at least an idea as to its character. It is obvious that many years must elapse before the orbit

can be obtained from the visual observations, and therefore it seems justifiable to include the data thus derived.

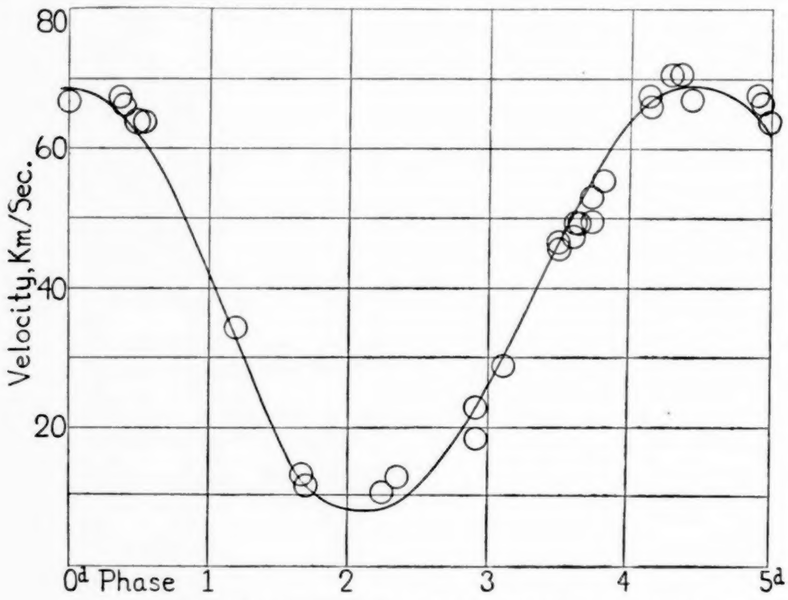

FIG. 1.—Velocity-curve of the spectroscopic binary μ Orionis

The curve in Figure 2 shows the change in the velocity of the center of mass, and indicates that a maximum was passed in 1908 and a minimum in 1913. The rough elements for the visual binary

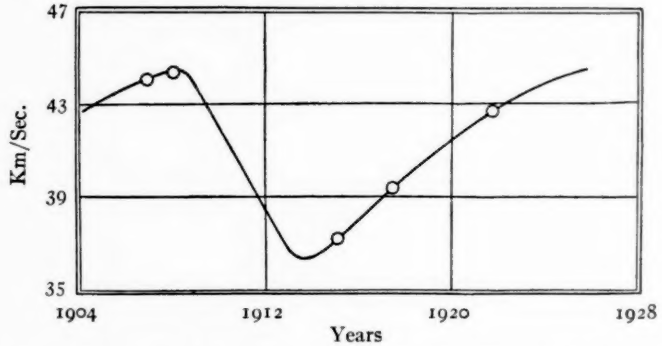

FIG. 2.—Velocity-curve of the center of mass of the spectroscopic binary μ Orionis

system would therefore be: $P=18$ years; $\gamma=+40.8$ km/sec.; $K=4.0$ km/sec.; $\omega=98°$; $e=0.6$; $T=1911.7$; $a \sin i=300$ million km.

Since the position angle has changed so little in 10 years, we may be justified in assuming that the orbit cannot be much inclined to the line of sight. We may therefore assume that $a \sin i$ in the visual orbit is not very different from a and equals about $0\overset{''}{.}3$. The spectroscopic orbit gives $a \sin i=300$ million km $=2.0$ astronomical units. This value refers to the orbit of the primary pair around the center of gravity of the visual binary system. Let m_1+m_2 be the total mass of the spectroscopic binary, and m_3 that of the Aitken visual companion. The value of $a \sin i$ in the visual orbit will be $\left(2.0\times\frac{m_1+m_2+m_3}{m_3}\right)$ astronomical units, and the parallax $=0\overset{''}{.}15\frac{m_3}{m_1+m_2+m_3}$. Mitchell's trigonometric parallax was $+0\overset{''}{.}036$. The hypothetical or dynamical parallax computed by Mr. Van Biesbroeck is $+0\overset{''}{.}018$. Taking the mean, we may assume for the absolute parallax of the star $+0\overset{''}{.}03$. Therefore $\frac{m_3}{m_1+m_2+m_3}=0.3$. By application of the Harmonic Law, we may finally derive a rough estimate of the total mass of the system of the three stars as five times that of the sun. Assuming that the mass ratio of the components of the spectroscopic binary $\frac{m_2}{m_1}$ is 0.8, which is the average for spectroscopic binaries of this class, the three masses composing this triple system would be as follows: $m_1=2.1\ \odot$; $m_2=1.7\ \odot$; $m_3=1.2\ \odot$. These results seem to be entirely reasonable.

It seems probable that the visual companion, already difficult, will soon be lost in the glare of the principal pair, so that the idea of the orbit gained above may have to serve as an approximation for some time to come. Visual observations will, undoubtedly, also establish the period and the inclination, the latter of course impossible of determination by spectroscopic procedure except as to sign. The sign of the inclination, however, ambiguous from visual observations, could be determined from spectroscopic observations. It may be that the difficulty of observing the visual companion will be so great that no better orbit of the binary system can be obtained than in this way.

The values of T in Table III differ by amounts which are several times larger than their probable errors. If the adopted value of the period is not quite correct, slightly different values of T would result, but in that case all these values, if plotted against the time, would fall on a straight line, which is not the case in fact. The cause is probably to be found in the light equation. This effect will produce a periodic oscillation of T, which will superimpose itself on the continuous change due to a possible error in P. Our present data are not sufficient to differentiate between the two effects. From the value of $a \sin i$, derived for the visual orbit, it appears that the total light equation may amount to as much as $0^d.03$, one-third of this being due to the orbit of the earth. When a better determination of the visual orbit becomes available, it will be necessary to correct all observations for light equation. It will then be possible to obtain a better value for the period of the spectroscopic binary.

It will evidently be necessary for us to continue the spectroscopic observations of this interesting star at least through the whole period of the variation of the velocity of the center of mass.

YERKES OBSERVATORY
April 17, 1924

Minor Contributions and Notes

NOTE ON THE DOUBLE STAR 9 ARGUS

In the *Astrophysical Journal*, **58**, 141, 1923, Otto Struve calls attention to the very interesting character of this system. He discusses the parallax of 9 Argus and comes to the conclusion that "it seems highly probable that the adopted value $0''.073$ is very close to the truth."

As pointed out by Struve, the values of the two modern photographic determinations by the trigonometric method are very discordant. Miller's measures of thirteen plates in five seasons from 1915.2 to 1917.2 give the relative parallax $+0''.121 \pm 0''.009$, while Mitchell's measures of twenty-two plates in six successive seasons from 1915.2 to 1917.8 furnish a much smaller relative parallax, $+0''.036 \pm 0''.003$. The only conclusion to draw is that one or the other of these values, or possibly both of them, are in error. In consequence of these discordances, it was decided by the writer some time ago to put this star again on the parallax program of the McCormick Observatory. The relative parallax from this second series of twenty-three plates in six successive seasons, from 1920.6 to 1923.3, is now available, with the value $+0''.028 \pm 0''.003$.

According to Aitken's ephemeris,[1] the first McCormick series of photographs was taken while the companion was passing through periastron, the position angles changing from 99° to 280°. During the second series, there was a change in position angle of only four degrees, from 290° to 294°, with no appreciable change in distance. The second series of plates was measured also by the writer. In order to change the conditions, a different set of comparison stars was used and also a different sector opening to cut down the brightness of the parallax star.

The mean of the two McCormick series gives the relative parallax of $+0''.032$ and the absolute parallax of $+0''.037 \pm 0''.006$.

[1] *Publications of the Lick Observatory*, **12**, 52.

Using this parallax and the period of 23.34 years, the mass of the system is 11.9 times that of the sun.

On account of the small changes in the orbit during the second series, these plates unfortunately could not be used to determine the ratio of the masses. Struve has found that he made a mistake in his formula[1] with the result that the ratio of the masses m/m_1 is 0.4 and not 0.6. The new value is therefore the same as that derived from the McCormick first series.

A word of caution might not come amiss to those astronomers who are using parallaxes in their investigations (and the list of such now includes practically every astronomer who is dealing with the stellar universe). Trigonometric parallaxes are the only ones directly observed and they are the only ones capable of giving the parallax of an individual star. All other methods of determining parallaxes are indirect methods based on certain assumptions, and at best they can furnish values of the *mean* parallax and *not* the direct parallax of the individual star. This criticism applies to spectroscopic, hypothetical parallaxes of all kinds, parallaxes from proper motions, from statistical tables, etc.

Russell,[2] Pannekoek,[3] and many others have called attention to the fact that the spectroscopic method cannot give the true parallax of an individual star unless the star is one of average mass. Pannekoek remarks (*loc. cit.*) that "the state of ionization in a stellar atmosphere, which determines the relative line intensity, is not physically connected with the luminosity of the star, but with the value of gravity at the surface of the star." He gives a simple formula connecting trigonometric and spectroscopic parallaxes of an individual star of known mass M, where M_0 is the mean mass of the group of stars used in constructing the reduction curves:

$$\frac{M}{M_0}=\left(\frac{\text{spectroscopic parallax}}{\text{trigonometric parallax}}\right)^2$$

Hence if the star has a mass greater than the average of stars of the particular type under investigation, the spectroscopic parallax will be too large. In *Eclipses of the Sun*, p. 315, the case is instanced

[1] *Astrophysical Journal*, **58**, 313, 1923. [2] *Ibid.*, **55**, 238, 1923.

[3] *Bulletin of the Astronomical Institutes of the Netherlands*, No. 19; *Observatory*, **46**, 304, 1923.

of the brilliant star Arcturus, of type Ko. The following values of the absolute parallax have been published:

Mean of Yale (heliometer) and Flint (mer. cir.)	$=+0\overset{''}{.}075$
Mean of Yerkes, Allegheny, and McCormick	$=\quad .092$
Mt. Wilson spectroscopic	$=\quad .158$
Norman Lockyer spectroscopic	$=\quad .145$
Victoria spectroscopic	$=\quad .100$
Harvard spectroscopic	$=+0.209$

The trigonometric parallaxes agree well among themselves, but the spectroscopic values do not. The former are systematically smaller than the latter. The only conclusion to draw is *not* that the trigonometric parallaxes are wrong but rather that Arcturus has a greater mass than the average star of type Ko, and it must therefore be a super-giant. For statistical purposes, when dealing with the mean of many stars, the spectroscopic method will furnish an entirely satisfactory series of values (always provided, of course, that these determinations are based on a sufficiently large number of trigonometric parallaxes). For an individual star, the case is entirely different, and the spectroscopic parallax can be correct only under the condition that the star under investigation departs but little in mass from the average.

The hypothetical method of Jackson and Furner has given excellent results. The parallaxes are derived from the assumption that each star of a binary system has the same mass as the sun, or that the mass of the system is twice that of the sun. Manifestly, this is an assumption and of necessity the hypothetical parallaxes of Jackson and Furner may differ greatly from the truth if the mass of the system differs greatly from twice that of the sun.

Unfortunately much of the reasoning carried out in astronomical investigations goes around in circles. For instance, if the mass of the system of 9 Argus is twice that of the sun, then the hypothetical parallax ($0\overset{''}{.}067$) is correct, and the spectroscopic value ($0\overset{''}{.}079$) must be near the truth and the McCormick trigonometric value ($0\overset{''}{.}037$) must be wrong. If the McCormick parallax is near the truth, the mass of the system of 9 Argus is 12 $\odot$, or six times the assumed value. Both the spectroscopic and hypothetical values are in this case manifestly too large. Assuming a mass of the

system of twelve times that of the sun, the hypothetical parallax must be reduced by $\sqrt[3]{6}$ and naturally becomes 0".037.

The question therefore resolves itself into the following: What is the mass of 9 Argus? The only feasible method of answering this question is by means of trigonometric parallaxes.

The writer would like to have other parallax observers put this star on their observing programs. A third series has been started at the McCormick Observatory. When it is completed, a third independent McCormick parallax will be derived, and all of the plates will be measured and discussed for the determination of trigonometric parallax, mass of the system, and mass-ratios.

S. A. MITCHELL

McCORMICK OBSERVATORY
December 1923

ABSORPTION OF MAGNESIUM VAPOR

ABSTRACT

Absorption of light by non-luminous *Mg* vapor was studied by heating pure metal in an iron tube to a temperature of 1200 C., and the line λ 4571 ($1S$-$2p_2$) was obtained distinctly as an absorption line. The line is of singular sharpness and appears only at comparatively high temperatures, while 2852 appears at low temperatures and is of remarkable width at high temperatures.

The experiments of Foote, Meggers, and Mohler on resonance and ionization potentials of metals of Group II show that these metals have two resonance potentials corresponding to the spectral lines $1S$–$2p_2$ and $1S$–$2P$. The fundamentally important line 4571 ($1S$–$2p_2$) of magnesium is the single-line spectrum emitted by the metal at the lower resonance potential. As such, according to the quantum theory of absorption, one would expect that this should be first absorbed by the non-luminous vapor of the metal. McLennan, using a long column of magnesium vapor, obtained only the reversal of the line 2852 ($1S$–$2P$), but could not detect the absorption of the fundamental line. King's work on tube-resistance furnace spectra shows that this line was obtained as a reversal line at low temperatures. But it might be contended that these spectra are not of purely thermal origin, and therefore the atoms of the metal might be in an excited state.

Recently the authors have started some experiments on the absorption of light by metallic vapors by heating the metal in each case in an iron tube in a coal furnace through which a powerful blast of air was maintained. In the course of these experiments the line 4571 of *Mg* was obtained distinctly as an absorption line. But it is worth noting that 2852 was not only absorbed at comparatively low temperatures, but also with increase of temperature it was remarkably widened, while the line 4571, which appeared only at high temperatures, was found to be one of singular sharpness even at the highest of temperatures used in these experiments, viz., 1200° C., nearly. If the remarkable broadening of the absorption lines is due to the Doppler effect and impact damping or the disturbance of inter-atomic fields by atomic collision, it is difficult to understand how the line 4571, which corresponds to the first resonance potential, does not increase in width with rise of temperature.

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A CORRECTION TO "THE DISTRIBUTION FUNCTIONS FOR STELLAR VELOCITY," MOUNT WILSON CONTRIBUTION NO. 272

Professor Grabowski, of Lemberg, has kindly called my attention to an error in equations (43) and (44) of "The Distribution Functions for Stellar Velocity" (*Astrophysical Journal*, **59**, 274, 1924).

The essential part of equation (40) can be written

$$P(\log T)=\frac{h_v}{\sqrt{\pi}}\sum_{\phi} E\Delta\phi \quad (\phi=5^\circ,\ 15^\circ,\ \ldots\ 85^\circ;\ \Delta\phi=\pi/18)$$

where E is defined by

$$\log E=-\text{Mod.}\ h_v^2(\log T-\log\cos\phi-\log \underline{v})^2+\log\cos\phi$$

This is the correct form of equation (43). The transition to the proper form of (44) is obvious.

The numerical data in Table III were obtained with correct formulae, however, and the conclusion as to the form of the function $P(\log T)\,d(\log T)$ is therefore unaffected by the error, which arose in condensing the equations for publication.

Frederick H. Seares

REVIEWS

Matter and Motion. By the late J. CLERK MAXWELL, sometime Professor of Experimental Physics in the University of Cambridge. Reprinted: with notes and appendices by SIR JOSEPH LARMOR. London: The Society for Promoting Christian Knowledge; New York: The Macmillan Co., 1920. 12mo. Pp. xv+163. Portrait and 19 Figs.

The much abused epithet "classical" is by common consent applied to the researches of Clerk Maxwell, and the Society for Promoting Christian Knowledge, which has so admirable a record of publication in the field of popular scientific exposition, is to be congratulated on its republication of this great thinker's elementary treatise on the laws of dynamic physics. It is not a rigid textbook for the working physicist, but a straightforward statement, suitable to the mentality of an undergraduate, of the mathematical conceptions under which are necessarily subsumed all rational investigation into the nature of physical phenomena. The man, who, during the nineteenth century, did more than any other to assemble the scattered thinking of his contemporaries upon physical matters and physical explanations under rational mathematical concepts, had the unusual and attractive ability to put his clear and comprehensive ideas into the most simple and direct language. A scholar and a gentleman, possessing literary ability of a high order, and the very opposite of the dryasdust professor of popular tradition, Clerk Maxwell could make the most abstract subject interesting; and the present little treatise (it is hardly more than a pamphlet) fully bears out his reputation on this score.

Clerk Maxwell, like so many other great seekers after physical truth, was a seer. He anticipated clearly the contemporary thought expressed most fully by that school which has culminated in Einstein. He thoroughly understood that physical science is the science of the motions of particles, which, when completely analyzed, are seen to possess only mathematical qualities and therefore to be devoid of the pseudo-metaphysical qualities of "substantiality" which the philosophers of the eighteenth century, following their old Greek predecessors, vainly strove to impart to them. It is with this relativistic conception that

Clerk Maxwell works his wonders, and leads the student, alternately stimulated and enthralled, from the simplest idea of the motion of a single hypothetical particle to the motions of the heavenly bodies and the system of Newton.

The book can be read by an undergraduate and by him followed intelligently. When he has finished it and grasped its ideas, he will see that the physical universe is one, both as a macrocosm and as a microcosm, and that the mathematical laws which enable us to understand the configurations, under the impress of energy of purely abstract material systems, are in themselves the explanations of the universe in its smallest as in its greatest manifestations.

Sir Joseph Larmor, the eminent holder of the Lucasian chair in Cambridge, has edited the treatise with some notes designed to bring before the contemporary reader some of the aspects of physico-mathematical investigation which were unavailable when Clerk Maxwell was alive. He has added two appendixes, one upon the relativity of motion and one upon the Rowan Hamilton principle of Least Action as deduced by that great Irish mathematician from Lagrange and here set forth in an admirably prepared short discussion designed to show the position of the principle in the determination of a general dynamic system. Sir Joseph is an opponent of the Theory of Relativity in its general form, and offers some stimulating notes upon the doctrine of absolute space and time, and its contradiction on quasi-metaphysical principles by the latest school of mathematical philosophers.

Clerk Maxwell died in 1878, yet it is only today that his influence can really be understood in its completeness and intensity. The man who fifty years ago grasped the identity of luminiferous with electromagnetic phenomena is the man without whom the present revolutionary doctrines of gravitation, of relative space, and of the boundless but finite universe would have lacked their principal justification.

William Braid White